Causality, Quantum Channels, and Mealy Machines

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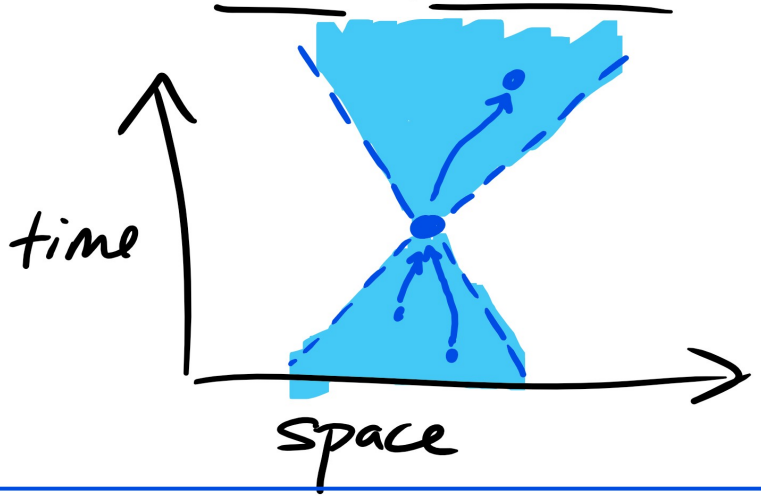
(<u>WITH</u> Juriaan Rot & Henning Basold)

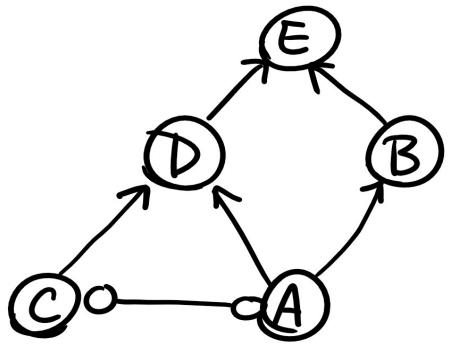
CAUSALITY appears in many forms:

In Physics

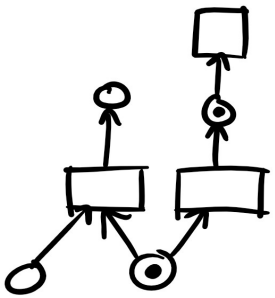


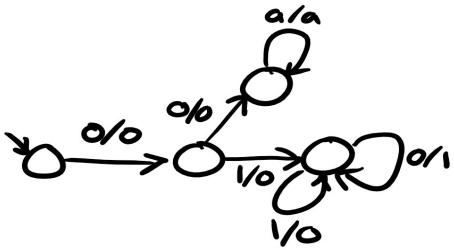
In statistics

$P(ABCDE) \rightsquigarrow$



In computation





THE GOAL : unify the way we <u>formulate</u> & <u>reason</u> <u>about</u> causal relationships

(between processes, events, programs, ...)

THE TOOL: Category theory (obviously ☺)

in particular, using

<u>PRECAUSAL</u> <u>CATEGORIES</u>

(:= SMC's + extra structure)

SYMMETRIC MONOIDAL CATEGORIES :=

* two compositions $\circ$ and $\otimes$

  sequential parallel

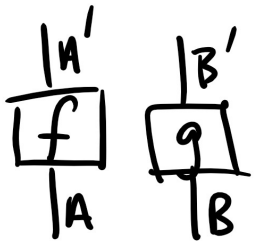
* swap maps $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$

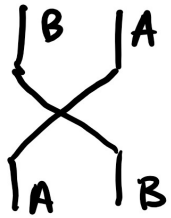
* + axioms...

STRING DIAGRAM NOTATION:

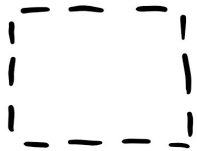
$f : A \to B \Rightarrow$ 

$g \circ f \Rightarrow$ 

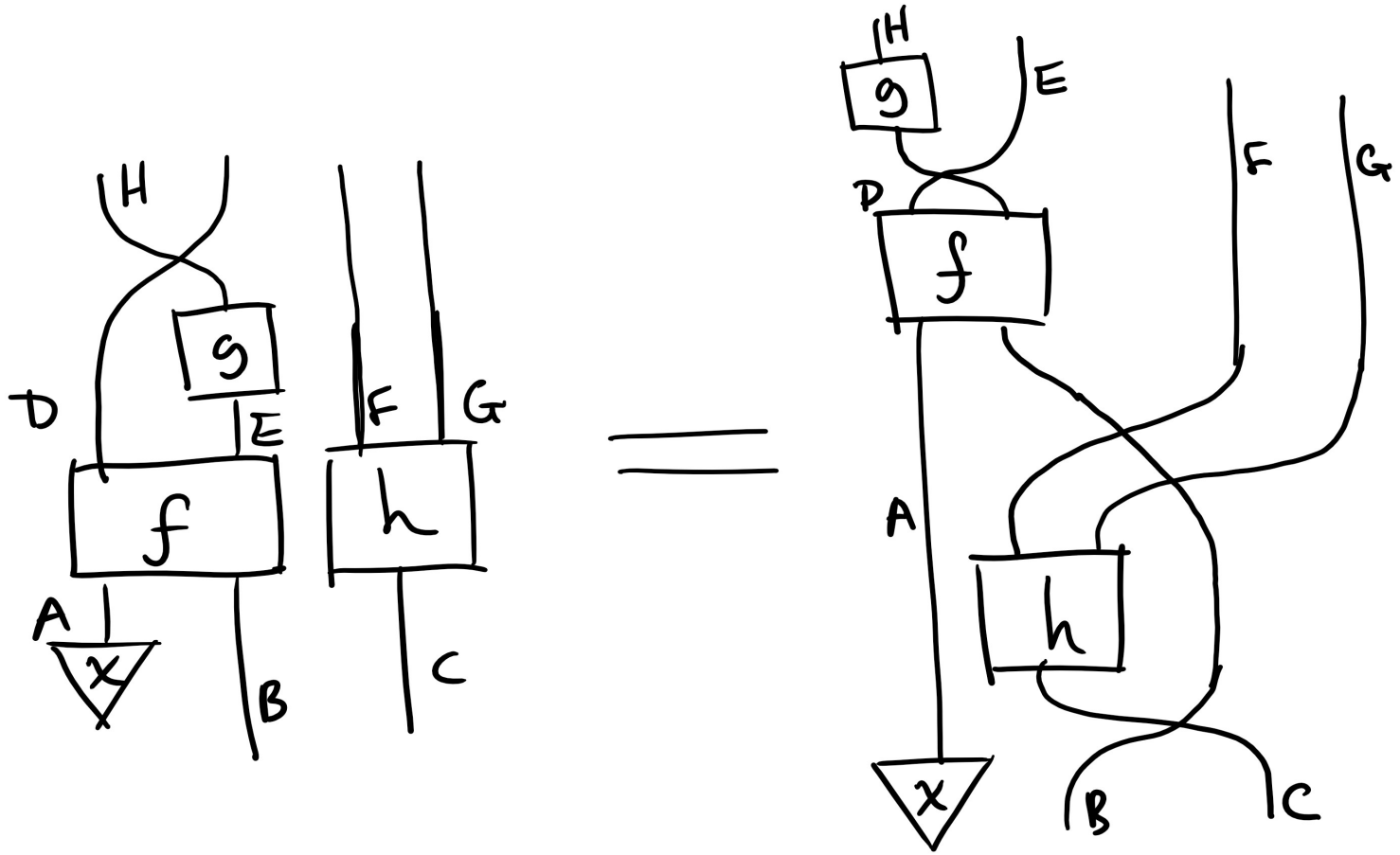
$f \otimes g \Rightarrow$ 

$\sigma_{A,B} \Rightarrow$ 

$1_A \Rightarrow$ 

I (or 1_I) $\Rightarrow$ 

SMC axioms $\Longleftrightarrow$ "only connectivity matters"



$$(\sigma_{D,H} \otimes 1_{F\otimes G}) \circ (1_D \otimes g \otimes 1_{F\otimes G}) \circ (f \otimes h) \circ (x \otimes 1_{B\otimes C}) \overset{\text{SMC}}{=\!=} (g \otimes 1_{E\otimes F\otimes G}) \circ (\sigma_{D,E} \otimes 1_{F,G}) \circ (1_A \otimes \sigma_{F\otimes G,B}) \circ (1_A \otimes h \otimes 1_B) \circ (x \otimes \sigma_{B,C})$$

Q: how do I get some causality into my SMC?

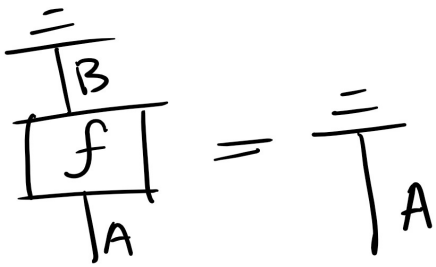
A: introduce a new process $d_A : A \to I$

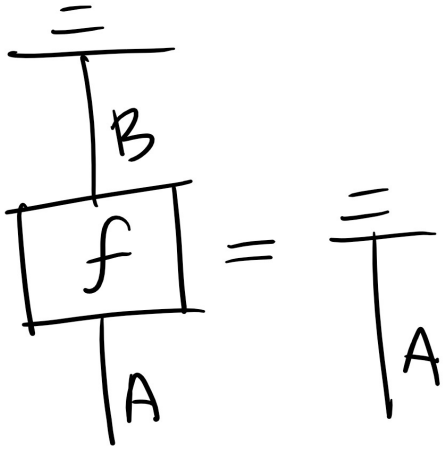
$d_A :=$ 

for discarding.

(my version)

<u>DEF</u> A process is <u>causal</u> if it satisfies:





"The only influence of f is on its output."

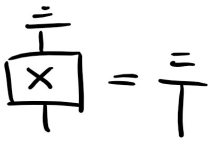
side-effect-free

information flows foreward in time

EXAMPLE:

$$\mathrm{Mat}(\mathbb{R}_+) \begin{cases} \underline{\text{ob}}\ n \in \mathbb{N} \\ \underline{\text{arr}}\ X : m \to n \text{ are } n \times m\ \mathbb{R}\text{-matrices} \\ \otimes : \text{tensor product}: Z_{ijkl} := X_{ij} Y_{kl} \end{cases}$$

$$\overline{\top}_k := (\underbrace{1\ 1\ \cdots\ 1}_{k})$$



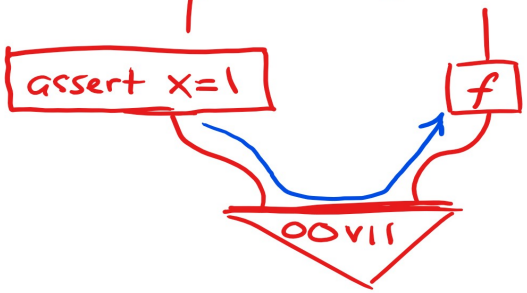
$\iff$ X is a <u>stochastic</u> matrix.

OTHER EXAMPLES:

* all functions (or morphisms in a category where I is terminal) are causal.

* in Rel causal means total

rules out "assertion" type processes



* in CPM, causal morphisms are quantum channels

ob: Hilbert spaces H, K
arr: completely positive maps $\mathcal{L}(H) \to \mathcal{L}(K)$.
$\otimes$: tensor product

(your version)

A function $f : A^\omega \to B^\omega$ is <u>causal</u>

if for all $n \in \mathbb{N}$, $\sigma \in A^\omega$:

$$\exists f' : A^n \to B^n . \quad f'(\sigma)\big|_n = \hat{f}(\sigma|_n)$$

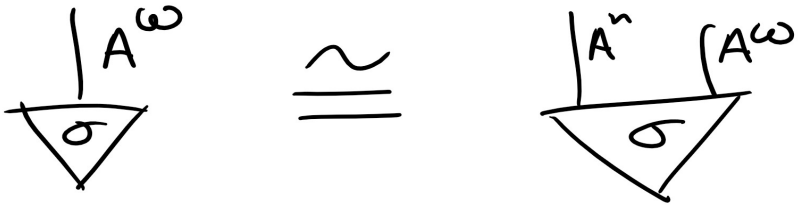
"The n-th output only depends on the first n inputs."

info flows foreword in time.

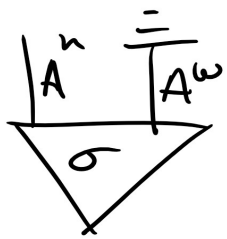
Q: can we picture "your" causality?

$$\forall n \in \mathbb{N}.\ \exists \text{ iso } \quad A^\omega \cong A^n \times A^\omega$$

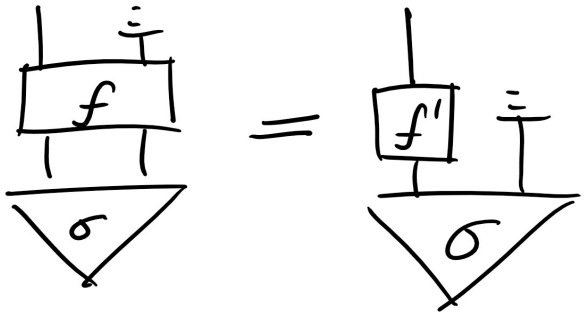
<u>RESTRICTION</u>:



$\sigma|_n \longleftrightarrow$

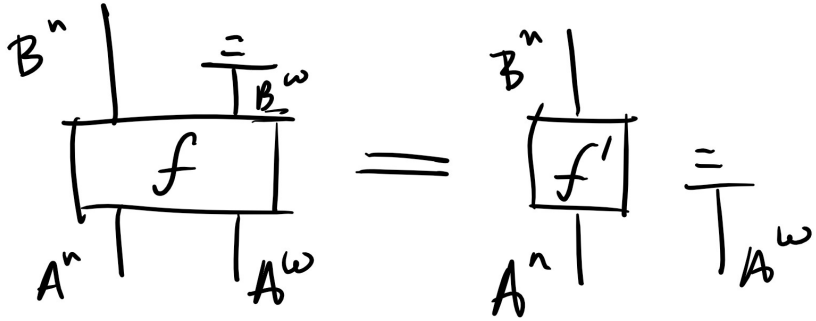


$$\exists f' .\ f(\sigma)\big|_n = f'(\sigma|_n) \quad \rightsquigarrow$$



SINCE THIS HOLDS FOR ALL σ, WE CAN WRITE:

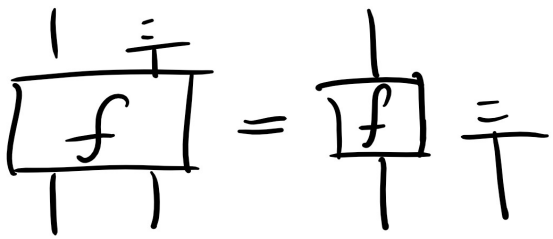
$$\forall n.\ \exists f' : A^n \to B^n .$$

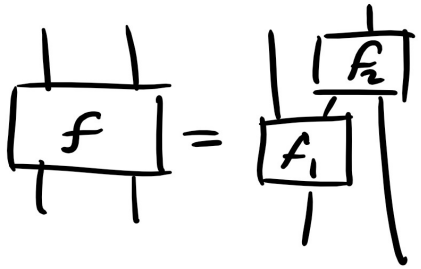


THIS EQ. HAS A NAME IN FOUNDATIONS OF PHYSICS, IT'S CALLED ONE-WAY NON-SIGNALLING.

An important <u>theorem</u> in quantum information theory (and the most important <u>axiom</u> for precausal categories...) is:

<u>ONE-WAY NON-SIGNALLING</u> $\Longrightarrow$ <u>SEMI-LOCALISABLE</u>

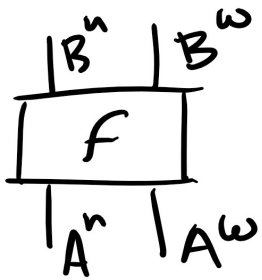
$\exists f!$ 

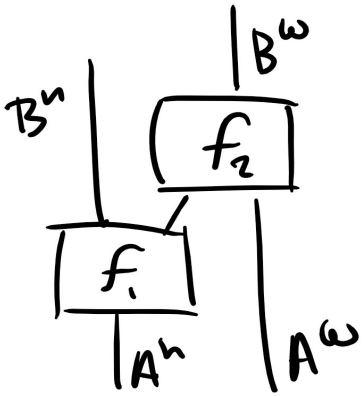
$\exists f_1, f_2.$ 

* CLASSICAL CASE: FOLLOWS FROM "CONDITIONAL DISINTEGRATION"
* QUANTUM CASE: Eggeling, Schlingemann, Werner '02

Q: Does this correspond to something for causal stream functions?

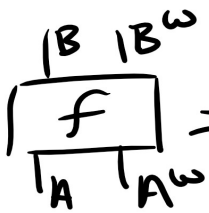
FIRST MASSAGE A BIT:

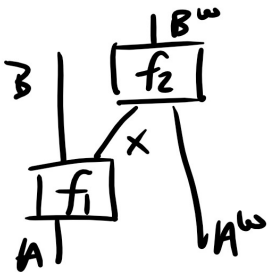
$\forall n \in \mathbb{N}.\ \exists f_1, f_2.$ 

$=$ 

SEMI-LOCALISABLE

$\Longleftrightarrow$ f IS ω-SEMI-LOC.

$:= \Big(\exists f_1, f_2.$ 

$=$ 

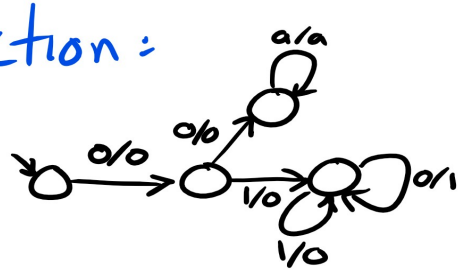
$\wedge\ \forall x.$ 

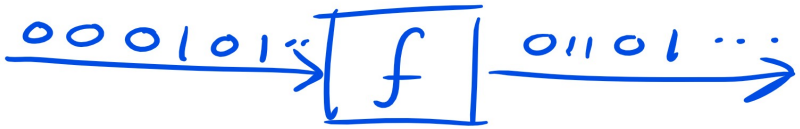
ω-SEMI-LOC.$\Big)$

MEALY MACHINES := coalgebras of

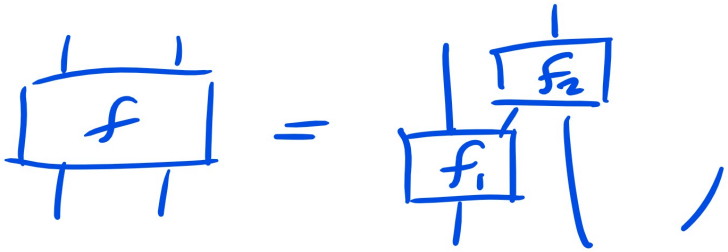
$$M_{A \to B}(-) := (B \times -)^A$$

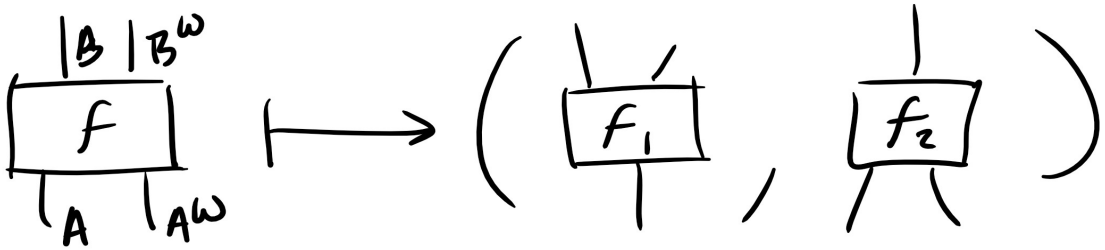
→ (det.) machine that implements a causal stream function:





THM CAUSAL FUNCTIONS ARE FINAL CO-ALGEBRA OF $\mathcal{M}(-)$

for 

$\hat{\phi} :=$ 

is (up to currying) the structure map.

CONCRETELY: THIS IS SYNTHESIS OF MEALY MACHINES FROM CAUS. FN'S USING STREAM DERIVATIVES.