The Ball Monad and its Metric Trace Semantics

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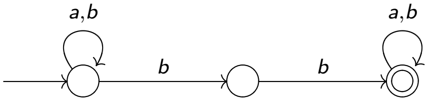
Overview

- Trace semantics in order-enriched Kleisli categories
- Definition of the ball monad
- Metric trace semantics for the ball monad

Non-deterministic automata

An example of a non-deterministic automaton:



This automaton recognizes the language

$$\{u \in \{a, b\}^* \mid u \text{ contains } bb \text{ as a subword}\}$$

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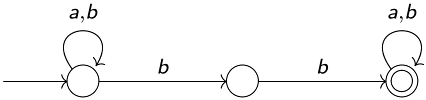

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Non-deterministic automata are coalgebras of the form

$$X \to 2 \times \mathcal{P}(X)^A \cong \mathcal{P}(1 + A \times X)$$

in the category **Sets**.

Non-deterministic automata

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This functor has no final coalgebra.
Can we nevertheless model the language recognized by the automaton via finality?

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A coalgebra $X \to \mathcal{P}(1 + A \times X)$ in **Sets** is the same as a coalgebra $X \to 1 + A \times X$ in **Rel**.
We will seek for a final coalgebra in **Rel**.

Non-deterministic automata

Theorem

The functor $1 + A \times X$ on **Rel** *has final coalgebra A^*.*
For each coalgebra $X \to 1 + A \times X$ in **Rel**, *this gives a relation $X \to A^*$, hence a map $X \to \mathcal{P}(A^*)$.*

Observe that A^* is also the initial algebra for $1 + A \times X$.

General situation

We consider coalgebras of the form $X \to TFX$, where:

- T is a monad on **Sets**.
- F is a functor on **Sets**.

Non-deterministic automata form an example, with $T = \mathcal{P}$ and $F(X) = 1 + A \times X$.

Coalgebras of the form $X \to TFX$ are morphisms $X \to FX$ in the Kleisli category $\mathcal{K}\ell(T)$. We need to "lift" the functor $F : \mathbf{Sets} \to \mathbf{Sets}$ to a functor $\bar{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$.

Lifting of functors

A distributive law $\lambda : FT \Rightarrow TF$ induces a lifting of $F : \mathbf{Sets} \to \mathbf{Sets}$ to $\bar{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$:

On objects: $\bar{F}(X) = F(X)$

On morphisms: $\bar{F}(\ X \xrightarrow{f} TY\ ) = (\ FX \xrightarrow{Ff} FTY \xrightarrow{\lambda} TFY\ )$

Coalgebras for TF in **Sets** correspond to coalgebras for $\bar{F}$ in $\mathcal{K}\ell(T)$.

Trace semantics in dcpo-enriched Kleisli categories

Theorem

Suppose that:

- $T(0) = 1$, *which implies that* $\mathcal{K}\ell(T)$ *has a zero object*
- $\mathcal{K}\ell(T)$ *is dcpo-enriched, in such a way that the zero maps are least elements in the Kleisli homsets*
- *The functor* F *has an initial algebra* $FA \xrightarrow{\alpha} A$
- *The functor* $\bar{F}$ *is locally monotone:*

$$f \leq g \Rightarrow \bar{F}(f) \leq \bar{F}(g)$$

Then $J(\alpha^{-1}) : A \to FA$ *is a final coalgebra for* $\bar{F}$ *in* $\mathcal{K}\ell(T)$.

Proof sketch

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{tr}_c} & A \\ \downarrow c & & \downarrow J(\alpha^{-1}) \\ FX & \xrightarrow{\bar{F}(\mathrm{tr}_c)} & FA \end{array}$$

Given a coalgebra $X \xrightarrow{c} FX$ in $\mathcal{K}\ell(T)$, we have to find a unique "trace map" tr_c making the diagram on the left commute. In other words, the operator $c; \bar{F}(-); J(\alpha) : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, A)$ should have a unique fixed point. The fixed point exists by the dcpo fixed point theorem. It is unique since α is an initial algebra.

Analogy between dcpos and metric spaces

Dcpo	Complete metric space
Continuous function	Contraction
Dcpo fixed point theorem	Banach fixed point theorem

Does the trace semantics for dcpo-enriched Kleisli categories have an analogue for metric spaces?

The Ball monad

Define the ball monad on objects as

$$\mathcal{B}(X) = \left\{ \varphi : X \to \mathbb{C} \,\middle|\, \sum_{x \in X} |\varphi(x)| \leq 1 \right\}$$

An element $\varphi \in \mathcal{B}(X)$ can also be written as a formal sum $\sum_i c_i x_i$ with $c_i \in \mathbb{C}$ and $x_i \in X$.
On morphisms,

$$\mathcal{B}(f)(\sum_i c_i x_i) = \sum_i c_i f(x_i)$$

The Kleisli category of the Ball monad

Let **Cms** be the category of complete metric spaces and non-expansive maps. $f : X \to Y$ is non-expansive if

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$

for all $x, x' \in X$.
The set $\mathcal{B}(Y)$ is a complete metric space with ℓ^1 metric

$$d(\varphi, \psi) = \sum_{y \in Y} |\varphi(y) - \psi(y)|$$

Hence the space of functions $\mathrm{Hom}_{\mathcal{K}\ell(\mathcal{B})}(X, Y) = X \to \mathcal{B}(Y)$ also forms a complete metric space with supremum metric. Pre- and post-composition are non-expansive, so $\mathcal{K}\ell(\mathcal{B})$ is enriched over **Cms**.

Metric trace semantics for the Ball monad

Theorem

Let F be a polynomial functor on **Sets** *with initial algebra $FA \xrightarrow{\alpha} A$, and let $\lambda : F\mathcal{B} \Rightarrow \mathcal{B}F$ be a distributive law. Then $J(\alpha^{-1}) : A \to FA$ is a final coalgebra for $\bar{F} : \mathcal{K}\ell(\mathcal{B}) \to \mathcal{K}\ell(\mathcal{B})$.*

Proof sketch

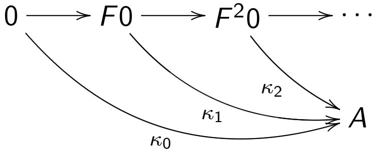
We wish to prove that the map

$$\mathrm{iter}_c = c; \bar{F}(-); J(\alpha) : \mathsf{Hom}(X, A) \to \mathsf{Hom}(X, A)$$

has a unique fixed point, using Banach's theorem. Unfortunately, iter_c is not a contraction. Therefore we modify the metric on $\mathsf{Hom}(X, A)$.

Proof sketch

The initial algebra A can be obtained as a colimit:



Define $\# : A \to \mathbb{N}$ by $\#a = \max\{n \mid a \notin \mathrm{im}(\kappa_n)\}$. Then define the metric on $\mathrm{Hom}(X, A)$ by

$$d(\varphi, \psi) = \sup_{x \in X} \sum_{a \in A} \frac{1}{2^{\#a}} \cdot |\varphi(x)(a) - \psi(x)(a)|$$

Apply Banach's fixed point theorem to find the unique fixed point.

Concluding remarks

- We have extended coalgebraic trace semantics to include the ball monad.
- This approach uses the fact that the Kleisli category of the ball monad is enriched over metric spaces, instead of partial orders.

Future research:

- Generalize this result to arbitrary metric-enriched Kleisli categories.
- Describe trace semantics for quantum computation.