Lax Extensions of Coalgebra Functors

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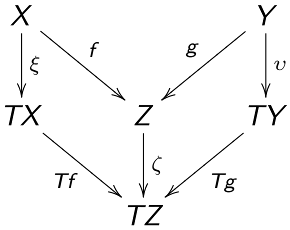
The Setting

We work with set based coalgebras.

Two states of coalgebras $\xi : X \to TX$ and $\upsilon : Y \to TY$ are *behaviorally equivalent* if there exists coalgebra morphisms that identify them.



Bisimilarity

A *relation lifting* L for T maps $R : X \nrightarrow Y$ to $LR : TX \nrightarrow TY$.

R is an *L-bisimulation* between $\xi : X \to TX$ and $\upsilon : Y \to TY$ if

$$(x, y) \in R \quad \text{implies} \quad (\xi(x), \upsilon(y)) \in LR.$$

Two states are *L-bisimilar* if an L-bisimulation connects them.

L captures behavioral equivalence if L-bisimilarity and behavioral equivalence coincide.

We assume that $L(R^\circ) = (LR)^\circ$

Example: Barr extension

The *Barr extension* $\overline{T}$ of T maps $R : X \nrightarrow Y$ to

$$\overline{T}R = \{(T\pi_X(\rho), T\pi_Y(\rho)) \mid \rho \in TR\}$$

where $\pi_X : R \to X$ and $\pi_Y : R \to Y$ are projections.

$\overline{T}$ captures behavioral equivalence if T preserves weak-pullbacks

Functors that do not preserve weak-pullbacks

The *neighborhood functor* $\mathcal{N} = \breve{\mathcal{P}}\breve{\mathcal{P}}$ where $\breve{\mathcal{P}}$ is the contravariant powerset functor.

The *monotone neighborhood functor* $\mathcal{M}$ is $\mathcal{N}$ restricted to upsets.

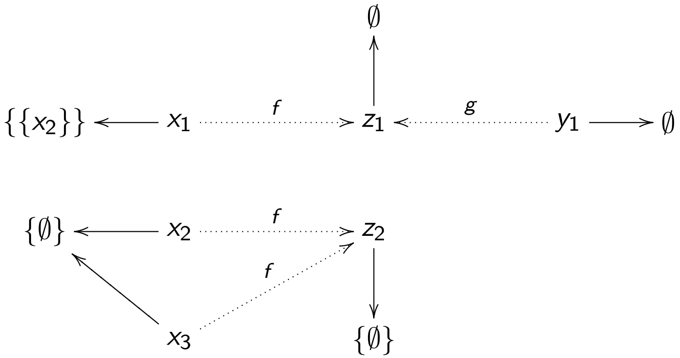
The *restricted powerset functor* $\mathcal{P}_n X = \{U \subseteq X \mid |U| < n\}$.

There are relation liftings $\widetilde{\mathcal{M}}$ for $\mathcal{M}$ and $\widetilde{\mathcal{P}_n}$ for $\mathcal{P}_n$ that capture behavioral equivalence.

Result

No relation lifting for $\mathcal{N}$ captures behavioral equivalence.

Proof:



Lax Extensions

L is a *lax extension* of T if for all compatible R, R', S and f:

1. $R' \subseteq R$ implies $LR' \subseteq LR$,
2. $LR\,;LS \subseteq L(R\,;S)$,
3. $Tf \subseteq Lf$.

A lax extension L *preserves diagonals* if it satisfies $Tf = Lf$.

Lax extension that preserves diagonals capture behavioral equivalence.

Theorem

A finitary functor T has a lax extension that preserves diagonals iff it has a separating set of monotone predicate liftings.

A predicate lifting λ for T is a natural transformation:

$$\lambda : T \Rightarrow \breve{\mathcal{P}}\breve{\mathcal{P}} = \mathcal{N}.$$

If λ is monotone its domain can be restricted:

$$\lambda : T \Rightarrow \mathcal{M}.$$

A set $\Lambda = \{\lambda : T \Rightarrow \mathcal{N} \mid \lambda \in \Lambda\}$ of predicate liftings is separating if $\{\lambda_X : TX \Rightarrow \mathcal{N}X \mid \lambda \in \Lambda\}$ is jointly injective for every set X.

Proof of Theorem

A finitary functor T has a lax extension that preserves diagonals iff it has a separating set of monotone predicate liftings.

Left-to-right uses the Moss liftings introduced by Kurz and Leal.

Right-to-left: For a set $\Lambda = \{\lambda : T \Rightarrow \mathcal{N} \mid \lambda \in \Lambda\}$ the initial lift $\widetilde{\mathcal{M}}^\Lambda$ of $\widetilde{\mathcal{M}}$ along Λ is defined on $R : X \nrightarrow Y$ as:

$$(\xi, \upsilon) \in \widetilde{\mathcal{M}}^\Lambda R \quad \text{iff} \quad (\lambda_X(\xi), \lambda_Y(\upsilon)) \in \widetilde{\mathcal{M}} R \text{ for all } \lambda \in \Lambda.$$