From Lawvere to Brandenburger-Keisler: interactive forms of diagonalization and self-reference

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Coalgebra and Reflexivity

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N.B. Return to caveats on last slide.

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This can be seen as a kind of many-person version of Russell's paradox.

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  This is $x \models \boxplus_a P$, where $\boxplus_a$ is the modality defined by

$$x \models \boxplus_a \phi \ \equiv \ \forall y. \, R_a(x, y) \Leftrightarrow \ y \models \phi.$$

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Our aim is to understand the general structures underlying this argument. Our first step is to recast their result as a *positive* one — a fixpoint lemma.

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Lemma (Basic Lemma)

From (1) and (2) we have:

$$p(x_0) \iff \exists y.[R_a(x_0, y) \wedge R_b(y, x_0)].$$

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(In fact $\neg q(x)$ is equivalent to their 'diagonal formula' D).

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- Where does this particular form "believes ... assumes ..." come from?
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- Under what circumstances can "sufficiently complete type spaces" be constructed? Coalgebra can be used here!

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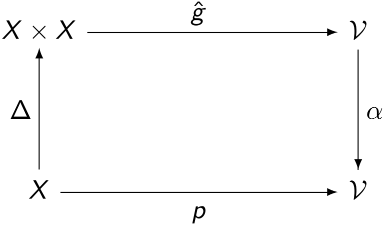

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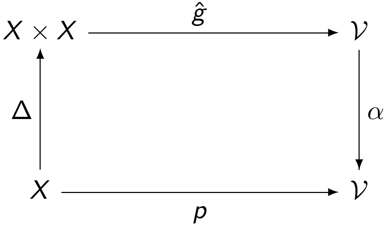

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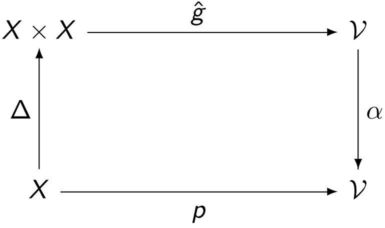

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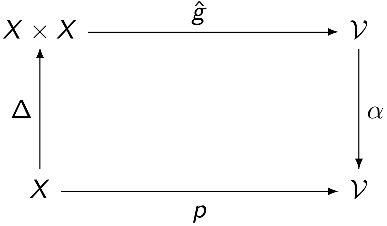

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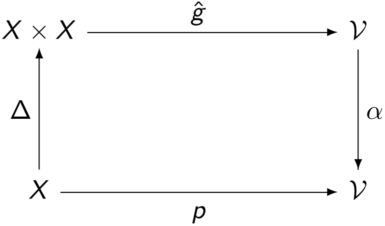
Some comments on the proof. (i) Constructive. (ii) Uses *two descriptions of p*.

The Fixpoint Lemma

Proposition

Suppose that $g : X \to \mathcal{V}^X$ *is surjective. Then every function* $\alpha : \mathcal{V} \to \mathcal{V}$ *has a fixpoint:* $v \in \mathcal{V}$ *such that* $\alpha(v) = v$.

Proof Define a predicate p by



There is $x \in X$ which represents p: then

$$p(x) = \alpha(\hat{g}(\Delta(x))) = \alpha(\hat{g}(x, x)) = \alpha(p(x))$$

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Some comments on the proof. (i) Constructive. (ii) Uses *two descriptions of* p. (iii) Since x represents p, $p(x)$ is (indirect) *self-application*.

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Suitably formulated, this is valid in any elementary topos.

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Russell's Paradox. Let $\mathcal{S}$ be a 'universe' (set) of sets. Let

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Such a predicate is given by the standard Russell set, which arises by applying the fixpoint lemma.

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Let $\mathcal{C}$ be a category with finite products.

(Lawvere) An arrow $f : A \times A \rightarrow V$ is *weakly point surjective* (wps) if for every $p : A \rightarrow V$ there is an $x : \mathbf{1} \rightarrow A$ such that, for all $y : \mathbf{1} \rightarrow A$:

$$p \circ y = f \circ \langle x, y \rangle : \mathbf{1} \rightarrow V$$

In this case, we say that *p is represented by x*.

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Let $\mathcal{C}$ be a category with finite products. If $f : A \times A \to V$ is weakly point surjective, then every endomorphism $\alpha : V \to V$ has a fixpoint $v : \mathbf{1} \to V$ such that $\alpha \circ v = v$.

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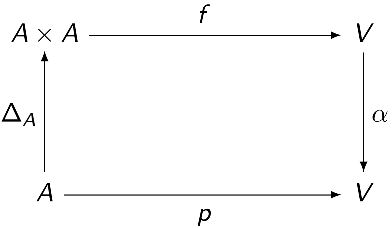

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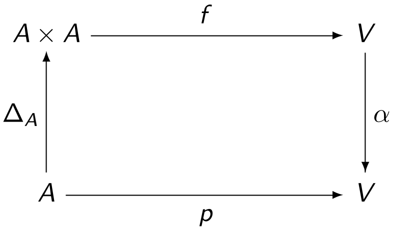

Suppose p is represented by $x : \mathbf{1} \to A$. Then

$$\begin{array}{rcll} p \circ x & = & \alpha \circ f \circ \Delta_A \circ x & \text{def of } p \\ & = & \alpha \circ f \circ \langle x, x \rangle & \text{diagonal} \\ & = & \alpha \circ p \circ x & x \text{ represents } p. \end{array}$$

so $p \circ x$ is a fixpoint of α. □

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The first step is to analyze exactly what logical resources are needed to carry through the BK argument.

Towards a categorical version of the BK argument

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This is a common fragment of intuitionistic and classical logic. It plays a core rôle in categorical logic.

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can be expressed as regular sequents as follows.

$$
\begin{array}{ll}
(A1) & R_a(c, y) \;\;\&\;\; R_b(y, x) \vdash_{\{x,y\}} p(x) \\
(A2) & R_a(c, y) \;\;\&\;\; p(x) \vdash_{\{x,y\}} R_b(y, x) \\
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Here (A1) and (A2) correspond to assumption (1) in the informal argument. We use c as a Skolem constant for x_0.

Formal Version of the Results

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The formal version of the Basic Lemma:

Lemma

From (A1)–(A3) we can infer the sequents:

$$p(c) \vdash q(c), \quad q(c) \vdash p(c)$$

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The formal version of the Fixpoint Lemma is now stated as follows:

Lemma

Under the assumptions (A1)–(A3), every definable unary propositional operator $O[\cdot]$ has a fixpoint, i.e. a sentence $S \equiv q(c)$ such that

$$S \vdash O[S], \qquad O[S] \vdash S.$$

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- If the propositional operator O is fixpoint-free, the result must be read contrapositively, as showing that the assumptions (A1)–(A3) lead to a contradiction. This will of course be the case if $O = \neg[\cdot]$ in classical logic. This yields exactly the BK argument.

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- In other contexts, this need not be the case. For example if the propositions (in categorical terms, the subobjects of the terminal object) form a complete lattice, and O is *monotone*, then by the Tarski-Knaster theorem there will indeed be a fixpoint. This offers a general setting for understanding why *positive logics*, in which all definable propositional operators are monotone, allow the paradoxes to be circumvented.

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Such a relation is *very weakly point surjective* (vwps) if for every subobject $P \rightarrowtail X$ there is $c : \mathbf{1} \to X$ such that:

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This weaker notion is sufficient to prove the Fixpoint Lemma.

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Naturality corresponds to *commuting with substitution*.

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If R is a vwps relation on X in a regular (even a lex) category, then every endomorphism of the subobject functor

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Then

$$\begin{aligned} [\![P(c)]\!] &= c^*(P) = c^*(\tau_X(\Delta_X^*(R)) = \tau_{\mathbf{1}}(c^* \circ \Delta_X^*(R)) = \tau_{\mathbf{1}}(\langle c, c\rangle^*(R)) \\ &= \tau_{\mathbf{1}}(c^*(P)) = \tau_{\mathbf{1}}([\![P(c)]\!]). \end{aligned}$$

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Our Basic Lemma can now be restated as follows:

Lemma

If R_a and R_b satisfy the BK assumptions (A1)–(A3), then R is vwps.

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Our Basic Lemma can now be restated as follows:

Lemma

If R_a and R_b satisfy the BK assumptions (A1)–(A3), then R is vwps.

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BK Reduced to Lawvere

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As an immediate Corollary, we obtain:

Lemma (BK Fixpoint Lemma)

If R_a and R_b satisfy the BK assumptions (A1)–(A3), then every endomorphism of the subobject functor has a fixpoint.

Multi-Agent Generalization

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A *multiagent belief structure* in a regular category is

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These assumptions can be written straightforwardly as regular sequents.

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There is also a kind of converse; see the paper in the Proceedings.

Using coalgebra to build assumption-complete type spaces

We are given strategy sets S_a, S_b for Alice and Bob respectively. We want to find sets of types T_a and T_b such that

$$T_a \cong \mathbf{P}(U_b), \qquad T_b \cong \mathbf{P}(U_a) \tag{1}$$

where $U_a = S_a \times T_a$ and $U_b = S_b \times T_b$ are the sets of states for Alice and Bob.

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Thus a state for Alice is a pair (s, t) where s is a strategy from her strategy-set and t is a type. Given an isomorphism $\alpha : T_a \xrightarrow{\cong} \mathbf{P}(U_b)$, we can define a relation $R_a : U_a \longrightarrow\!\!\!\!\!\!\!+\;\; U_b$ by:

$$R_a((s,t),(s',t')) \;\equiv\; (s',t') \in \alpha(t).$$

Note that (s, t) **assumes** $\alpha(t)$. Because α is an isomorphism, the belief model (U_a, U_b, R_a, R_b) is automatically assumption complete with respect to $\mathbf{P}(U_a)$ and $\mathbf{P}(U_b)$.

General Formulation

Suppose that we have a category $\mathcal{C}$, which we assume to have finite products, and a functor $\mathbf{P} : \mathcal{C} \to \mathcal{C}$. We are given objects S_a and S_b in $\mathcal{C}$. Hence we can define functors $F_a, F_b : \mathcal{C} \to \mathcal{C}$:

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Now we define a functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ on the product category:

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To ask for a pair of isomorphisms as in (1) is to ask for a *fixpoint* of the functor F: an object of $\mathcal{C} \times \mathcal{C}$ (hence a pair of objects of $\mathcal{C}$, (T_a, T_b)) such that

$$(T_a, T_b) \cong F(T_a, T_b).$$

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Standard results allow us to lift one-person to two- (or multi-)agent constructions. Suppose we have endofunctors $G_1, G_2 : \mathcal{C} \rightarrow \mathcal{C}$. We can define a functor

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$$(\mathbf{1}, \mathbf{1}) \leftarrow (G_1(\mathbf{1}), G_2(\mathbf{1})) \leftarrow (G_1(G_2(\mathbf{1})), G_2(G_1(\mathbf{1})) \leftarrow$$

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This 'symmetric feedback' is directly analogous to constructions which arise in Geometry of Interaction and the Int construction. It is suggestive of a compositional structure for interactive belief models.

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These models are all closed under conjunction, disjunction, existential and universal quantification, and constructions corresponding to the **assumes** and **believes** modalities (in the powerdomain cases, with some order-theoretic saturation).
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They are not, of course, closed under negation!

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- Is there reflexivity in physics?
- What is the scope of of interactive versions of logical and mathematical phenomena which have previously only been studied in 'one-person' versions?