

Pointwise Extensions of GSOS-Defined Operations

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Streams and Mealy machines

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Final coalgebra: B^ω

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Final coalgebra: $\Gamma = \{m : A^\omega \rightarrow B^\omega \mid m \text{ causal } \}$

$$m(a_1, a_2, a_3, \dots) = b_1, b_2, b_3, \dots$$

b_i depends only on a_1, \dots, a_i

Some operations on streams

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Pointwise extensions

For any operation $o : (B^\omega)^n \rightarrow B^\omega$

the pointwise extension $\bar{o} : \Gamma^n \rightarrow \Gamma$ is:

$$\bar{o}(m_1, \dots, m_n)(\sigma) = o(m_1(\sigma), \dots, m_n(\sigma))$$

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Q: Does \bar{o} come from a dist. law if o does?

A naive approach

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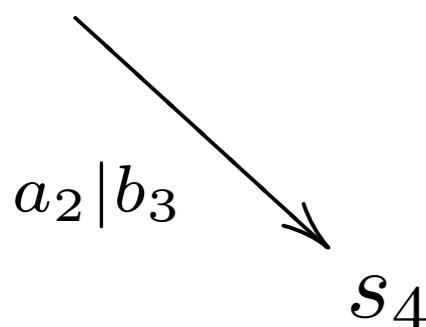
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$$t_1 \xrightarrow{a_1|c_1} t_2 \xrightarrow{a_2|c_2} t_3$$

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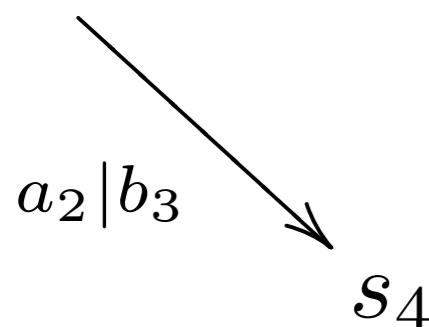
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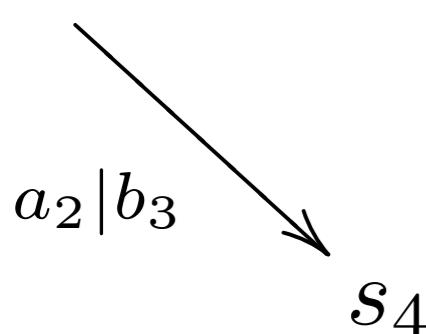
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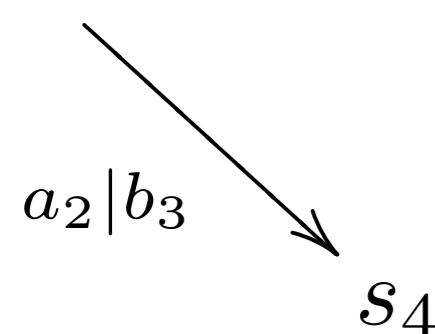
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but:

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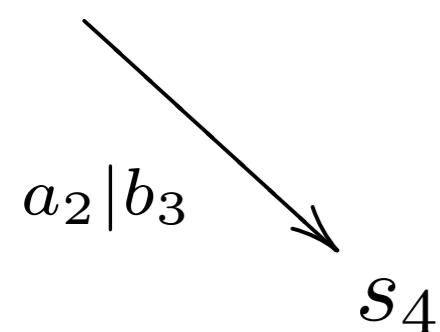
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Buffer operations

Solution: use auxiliary operators $a \triangleright -$, for $a \in A$

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Fact: This defines the pointwise extension of \boxplus .

Fact: It also works for any GSOS-defined operation,
and even for arbitrary F .

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2. Any $\lambda : \Sigma(\text{Id} \times F) \Rightarrow FT_\Sigma$

can be pointwise extended to

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where $\Sigma'X = \Sigma X + A \times X$