Strong relative monads

Tarmo Uustalu, Institute of Cybernetics, Tallinn

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Motivation and contribution

- In programming language theory, we use structures like (strong) monads, (monoidal) comonads, arrows to structure syntax and semantics.
- Some natural structures fail to be monads as if for the only reason that the underlying functor is not an endofunctor.
- E.g., untyped/typed lambda calculus syntax (over finite contexts), finite-dimensional vector spaces etc.

- In FoSSaCS 2010, we defined and studied a relative monads as a generalization of monads.
- Here: strong relative monads.

Relative monads

- Given a category \mathbb{C} and another category \mathbb{J} with a functor $J \in \mathbb{J} \to \mathbb{C}$.
- A relative monad is given by
 - an object function $T \in |\mathbb{J}| \to |\mathbb{C}|$,
 - for any object $X \in |\mathbb{J}|$, a map $\eta_X \in \mathbb{C}(\mathbb{J}X, \mathcal{T}X)$ (unit),
 - for any objects X, Y ∈ |J| and map k ∈ C(JX, TY), a map k* ∈ C(TX, TY) (Kleisli extension)

satisfying

- for any $X, Y \in |\mathbb{J}|$, $k \in \mathbb{C}(\mathbb{J}X, T|Y)$, $k^* \circ \eta_X = k$,
- for any $X \in |\mathbb{J}|$, $\eta_X^* = \operatorname{id}_{TX} \in \mathbb{C}(TX, TX)$,
- for any $X, Y, Z \in |\mathbb{J}|$, $k \in \mathbb{C}(\mathbb{J}X, TY)$, $\ell \in \mathbb{C}(\mathbb{J}Y, TZ)$, $(\ell^* \circ k)^* = \ell^* \circ k^* \in \mathbb{C}(TX, TZ)$.
- T is functorial with T f = (η ∘ J f)*; η and (−)* are natural.

Relative monads (ctd)

- Ordinary monads arise as the special case where $\mathbb{J}=_{\mathrm{df}}\mathbb{C},$ $J=_{\mathrm{df}}\mathrm{Id}_{\mathbb{C}}.$
- Can define relative adjunctions between $J \in \mathbb{J} \to \mathbb{C}$ and \mathbb{D} .
- Every relative adjunction gives rise to a relative monad.
- Every relative monad resolves into a relative adjunction in at least two ways, the Kleisli and E-M adjunctions, which are its initial and final resolutions.
- If Lan_J ∈ [J, C] → [C, C] exists, then [J, C] has a lax monoidal structure and a relative monad on J is a lax monoid in it.
- If further conditions on J hold (in particular, J is fully faithful), then [J, C] is (properly) monoidal and a relative monad on J is a (proper) monoid in it.

Example

- Given a semiring $(R, 0, +, 1, \times)$.
- Let $\mathbb{J} =_{\mathrm{df}} \mathbb{F}$, $\mathbb{C} =_{\mathrm{df}} \mathbf{Set}$, $J = \mathsf{the}$ inclusion.
- Define
 - a set mapping $T \in \mathbb{F} \to \mathbf{Set}$ by $T \ m =_{\mathrm{df}} J \ m \to R$,
 - for any $m \in |\mathbb{F}|$, a function $\eta_m \in J \ m \to T \ m$ by $\eta_m (i \in m) =_{\mathrm{df}} \lambda j \in m$. if i = j then $1 \mathrm{else} 0$
 - for any $m, n \in |\mathbb{F}|$, $A \in J m \to T n$, a function $A^* \in T m \to T n$ by $A^* x =_{df} \lambda j \in n$. $\sum_{i \in m} x i \times A i j$

T m is the space of m-dimensional vectors, η_m is the diagonal $(m \times m)$ -matrix, and $A^* x$ is the product of matrix A with a vector x.

- $(T, \eta, (-)^*)$ is a relative monad.
- KI(*T*) is the category of finite-dimensional vector spaces and linear transformations.

Weak arrows

- \bullet Given a category $\mathbb J,$ a *weak arrow* on $\mathbb J$ is given by
 - an object function $R \in |\mathbb{J}| \times |\mathbb{J}| \to \mathbf{Set}$,
 - for any objects $X, Y \in |\mathbb{J}|$, a function pure $\in \mathbb{J}(X, Y) \to R(X, Y)$,
 - for any $X, Y, Z \in |\mathbb{J}|$, a function (\ll) $\in R(Y, Z) \times R(X, Y) \rightarrow R(X, Z)$

satisfying

- pure $(g \circ f) = pure g \ll pure f$,
- *r* ≪ pure id = *r*,
- pure id $\ll r = r$,
- $t \ll (s \ll r) = (t \ll s) \ll r$.
- *R* extends to a functor J^{op} × J → Set (an endoprofunctor on J); pure and ≪ are natural.

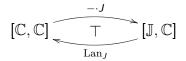
Weak arrows = relative monads on Yoneda

- Assume \mathbb{J} is small. Let $\mathbb{C} =_{df} [\mathbb{J}^{op}, \mathbf{Set}]$, $J = \mathbf{Y}$ (the Yoneda embedding).
- A weak arrow on \mathbb{J} is a functor $R \in \mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbf{Set}$ with structure.
- This is the same as a functor $T \in \mathbb{J} \to [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$ with structure, in fact, a relative monad on \mathbf{Y} .

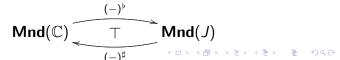
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Monads vs relative monads

- Given any \mathbb{C}, \mathbb{J} and $J \in \mathbb{J} \to \mathbb{C}$.
- If T is a monad on C, then T^b =_{df} T ⋅ J is a relative monad on J.
- If J is well-behaved, then
 If T is a relative monad on J, then T[♯] =_{df} Lan_J T is a monad on C.
- The adjunction



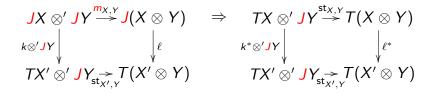
lifts to an adjunction (a coreflection, if we require that J is fully-faithful)



Strong relative monads

- Given a monoidal categories (J, I, ⊗), (C, I', ⊗') and a monoidal functor (J, e, m) between them.
- A strong relative monad is a relative monad (T, η, (−)*) and, for any X, Y ∈ |J|, a map st_{X,Y} ∈ C(TX ⊗' JY, T(X ⊗ Y)), natural in X, Y, with T, η, (−)* strong wrt st, so that

$$\begin{array}{c|c} TX \otimes' I' \xrightarrow{TX \otimes' e} TX \otimes' JI \xrightarrow{\operatorname{st}_{X,I}} T(X \otimes I) \\ & & & \downarrow^{\rho_X} \\ & & & \downarrow^{\rho_X} \\ TX = TX \end{array}$$



Arrows

- Given a (small) monoidal category (\mathbb{J}, I, \otimes) .
- An arrow on (J, I, ⊗) is a weak arrow (R, pure, ≪) on J with, for any X, Y, Z ∈ |J|, a map first_{X,Y,Z} ∈ R(X, Y) → R(X ⊗ Z, Y ⊗ Z) satisfying

- pure (id $\otimes f$) \ll first r = first $r \ll$ pure id $\otimes f$)
- pure $ho \lll first r = r \lll pure
 ho$
- pure $\alpha \ll \text{first}(\text{first } r) = \text{first} r \ll \text{pure } \alpha$
- first (pure f) = pure ($f \otimes id$)
- first $(s \ll r) =$ first $s \ll$ first r

• first_{X,Y,Z} is natural in X, Y, dinatural in Z.

Arrows = strong relative monads on Yoneda

- Let \mathbb{J} be small, take $\mathbb{C} =_{df} [\mathbb{J}^{op}, \mathbf{Set}]$, $J =_{df} \mathbf{Y}$ (Yoneda on \mathbb{J}).
- A monoidal structure (1, \otimes) on $\mathbb J$ induces one on $\mathbb C$ via
 - $I'Z =_{\mathrm{df}} \mathbb{J}(Z, I),$
 - $(F \otimes' G) Z =_{df} \int^{X, Y \in |\mathbb{J}|} \mathbb{J}(Z, X \otimes Y) \times (FX \times GY)$ (the Day convolution)
- Y becomes a monoidal functor.
- Consider a strong relative monad $(T, \eta, (-)^*, st)$.
- We have

$$\begin{array}{l} (T X \otimes' \mathbf{Y} Y)Z \\ = \int^{X',Y' \in |\mathbb{J}|} \mathbb{J}(Z,X' \otimes Y') \times (T X X' \times \mathbb{J}(Y',Y)) \\ \cong \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z,X' \otimes Y) \times T X X' \end{array}$$

Hence

 $(\mathsf{st}_{X,Y})_Z \in \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y) \times T X X' \to T (X \otimes Y) Z$ which is equivalent to having a map first_{X',X,Y} \in T X X' \to T (X \otimes Y) (X' \otimes Y), \text{ for a product of } \mathbb{F}

Arrows = strong monads in **Prof**

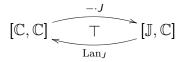
- Cf. Jacobs et al. (2006), Asada (2010)
- Arrows on a (small) category \mathbb{J} are monoids in the category on the endoprofunctors on \mathbb{J} .
- Arrows are monads in the bicategory **Prof** of (small) categories and profunctors.

Strong monads vs strong relative monads

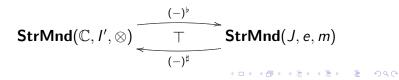
- If T is a strong monad on $(\mathbb{C}, I', \otimes')$, then $T^{\flat} =_{df} T \cdot J$ is a strong relative monad on (J, e, m).
- If J is well-behaved, then
 if T is a strong relative monad on (J, e, m), then

 $T^{\sharp} =_{df} \operatorname{Lan}_{I} T$ is a strong monad on (C, I', \otimes') .

The adjunction



lifts to an adjunction



Conclusions

- Adding strength to relative monads is not difficult.
- Key idea: J must be a monoidal functor.
- Arrows become strong relative monads, are hence a natural structure.

Hughes, Paterson got the axioms right without deriving arrows as an instance of something more general!

Future work

- Alternative descriptions of strong relative monads.
- Formalization in Agda.
- Arrow metalanguage (cf. Lindley, Wadler, Yallop 2010).

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