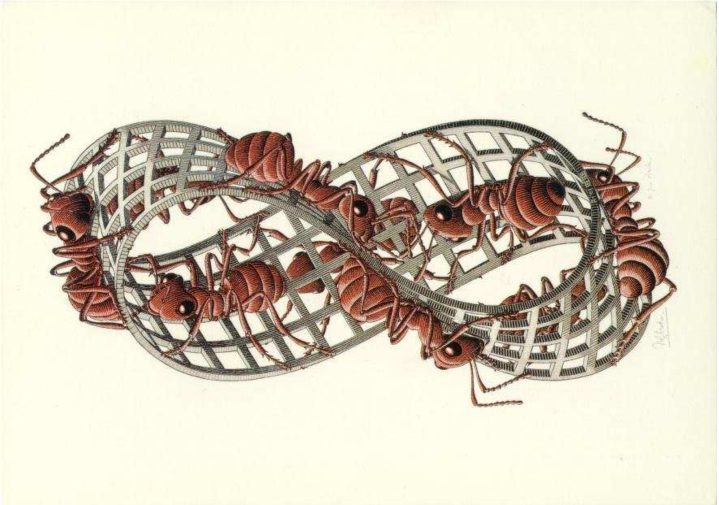
Coalgebras in Type Theory

Venanzio Capretta

CMCS 2010, Paphos, Cyprus

Corecursive Equations



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Streams: infinite sequence over a domain D, $\mathbb{S}_D$.

$$\mathsf{nat} = 0 :: 1 :: 2 :: 3 :: 4 :: 5 :: 6 :: \cdots : \mathbb{S}_{\mathbb{N}}$$
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$${}^{4\mathfrak{t}}\mathsf{nat} = 4 :: 5 :: 6 :: 7 :: 8 :: 9 :: 10 :: \cdots$$
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Corecursive equations on streams: [Rutten 2007]

$$\mathsf{nat} = 0 :: \mathsf{nat} + 1 \qquad \mathsf{fib} = 0 :: \mathsf{fib} + (1 :: \mathsf{fib})$$
$$(x :: s_1) \ltimes s_2 = x :: s_2 \ltimes s1$$
$$\mathsf{even}\,(x :: s) = x :: \mathsf{odd}\ s \qquad \mathsf{odd}\,(x :: s) = \mathsf{even}\,s$$

Harder corecursive equations

Equations that are more difficult to solve [Zantema 2009]

Three functions of type $\mathbb{S} \to \mathbb{S}$:

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Puzzle: Find equation $f\ s = C[s, f]$ that generates:

$$\begin{aligned} f\ \mathsf{nat} = \ & 0 :: 0 :: 1 :: 0 :: 2 :: 1 :: 3 :: 0 :: 4 :: 2 :: 5 :: 1 :: 6 :: 3 :: 7 :: 0 :: 8 \\ & :: 4 :: 9 :: 2 :: 10 :: 5 :: 11 :: 1 :: 12 :: 6 :: 13 :: 3 :: 14 :: 7 :: 15 \\ & :: 0 :: 16 :: 8 :: 17 :: 4 :: 18 :: 9 :: 19 :: 2 :: 20 :: 10 :: 21 :: 5 :: 22 \\ & :: 11 :: 23 :: 1 :: 24 :: 12 :: 25 :: 6 :: 26 :: 13 :: 27 :: 3 :: 28 :: 14 \\ & :: 29 :: 7 :: 30 :: 15 :: 31 :: 0 :: 32 :: 16 :: 33 :: 8 :: 34 :: 17 :: 35 \\ & :: 4 :: 36 :: 18 :: 37 :: 9 :: 38 :: \cdots \end{aligned}$$

Images of streams of Booleans [Zantema]

The Boolean Fibonacci stream:

$$f\,(0 :: s) = 0 :: 1 :: f\,s \qquad \mathrm{bfib} = f\,\mathrm{bfib}$$
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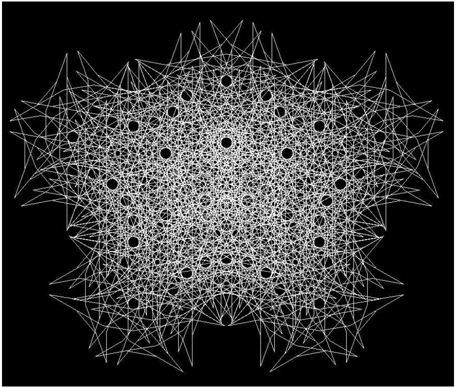

bfib = 0 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : 0 : 0
:1 : 0 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : 0 : 0 : 1
:0 : 1 : 0 : 0 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : 0
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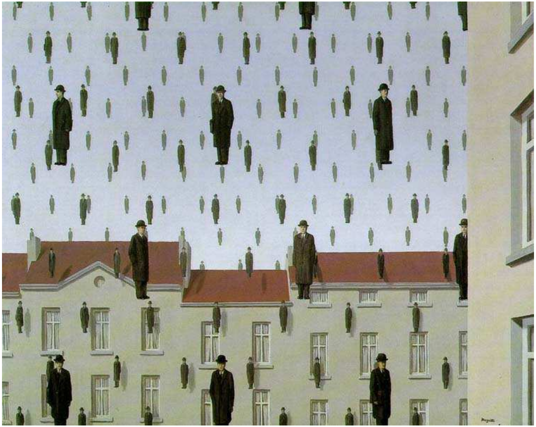
CoInductive Types



CoInductive Types: [Hagino 1987, Aczel & Mendler 1989]

Type-theoretic implementation of final coalgebras.

CoInductive Definitions

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Type-theoretic implementation of final coalgebras.

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  $(::) : D \to \mathbb{S}_D \to \mathbb{S}_D$

Final Coalgebra:
$\langle {}^{\mathfrak{h}}\_, {}^{\mathfrak{t}}\_ \rangle : \mathbb{S}_D \to D \times \mathbb{S}_D$

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$: \mathbb{T}_{A,B} \to B + (A \to \mathbb{T}_{A,B})$

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$(::)$, leaf and node are constructors

Guardedness by constructors: [Coquand 1993]

A corecursive equation has a unique solution if all recursive calls occur only directly under constructor applications.

Example of guarded definition

Puzzle: 0 : 0 : 1 : 0 : 2 : 1 : 3 : 0 : 4 : 2 : 5 : 1 : 6 : 3 : 7 : 0 : 8 : 4 : 9 : 2 : 10 : 5 : 11 : 1 : 12 : 6 : 13 : 3 : 14 : 7 : 15

Definitions are accepted if they satisfy the guardedness condition [Gimènez 1998]

$$
\begin{aligned}
&\mathsf{fguard} : \mathbb{N} \to \mathbb{S} \\
&\mathsf{fguard}\, n = \mathbf{case}\, \mathsf{mod}(n, 3) \\
&\begin{cases}
0 \mapsto \mathsf{nat} \\
1 \mapsto n :: (n-1) :: \mathsf{fguard}\,(n+1) \\
2 \mapsto n :: \mathsf{map}\,(2 \cdot -)\,(\mathsf{fguard}\,(2 \cdot n))
\end{cases}
\end{aligned}
$$

No recursive calls.
Recursive calls under two constructors.
Map *filters* the constructors.

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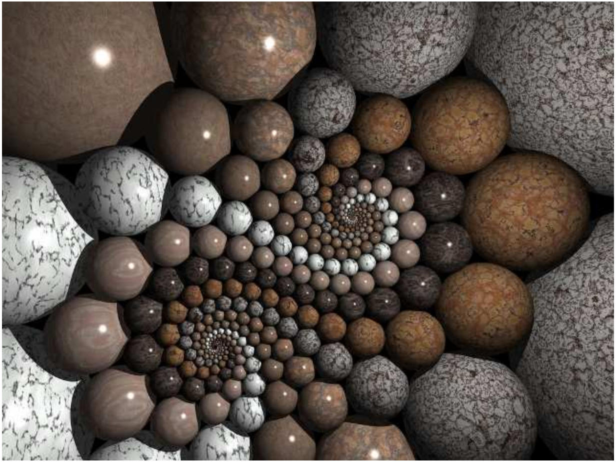
- Unicity by Bisimulation [VC 2010]

Bisimulations



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The Coinduction principle:

$$x_1 \sim x_2 \Rightarrow x_1 = x_2.$$

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$$\mathsf{nodesim} : (f_1, f_2 : A \to \mathbb{T})(\forall a. f_1\ a \approx f_2\ a) \to (\mathsf{node}\ f_1) \approx (\mathsf{node}\ f_2)$$

Bisimulation and Unicity of Solutions

Unicity of solutions for the equation:

$$\chi : \mathbb{S} \to \mathbb{S}$$
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Coalgebras in Type Theory

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Corecursive Equations

CoInductive Types

Bisimulations

Constructive Infinity

Tabulations

General Recursion

Non-Standard Type Theory

Mixing Induction and Coinduction

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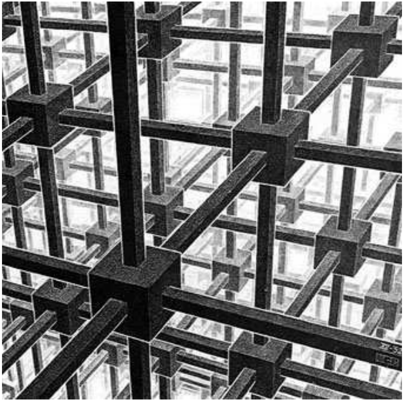
By the coinduction principle and R0, $\chi_1 = \chi_2$.

Constructive Infinity



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Brouwer's Continuity (In Functional Programming terms):
Given a function $f : \mathbb{S}_{\mathbb{N}} \to \mathbb{N}$,
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&\mathsf{counterexample} : (\mathbb{S}_{\mathbb{B}} \to \mathbb{B}) \to \mathbb{S}_{\mathbb{B}} \\
&\mathsf{counterexample}\, f = \ \ \mathbf{if}\ (\mathsf{allb}\, f_t) \\
&\qquad\qquad\qquad\qquad\quad \mathbf{then}\ (\mathsf{false} :: \mathsf{counterexample}\, f_f) \\
&\qquad\qquad\qquad\qquad\quad \mathbf{else}\ (\mathsf{true} :: \mathsf{counterexample}\, f_t) \\
&\qquad \mathbf{where}\ \ f_t = \lambda s. f\,(\mathsf{true} :: s) \\
&\qquad\qquad\qquad\ \ f_f = \lambda s. f\,(\mathsf{false} :: s)
\end{aligned}
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Integers in binary representation: $[\mathbb{B}]$.

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A function $f : [\mathbb{B}] \to A$ can be represented by a tree:

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Tabulation of the function:

$$\begin{aligned}&\mathrm{tabulate} : ([\mathbb{B}] \to A) \to \mathbb{T}_A\\ &\mathrm{tabulate}\, f = \mathrm{node}\,(f\,[\,])\,(\mathrm{tabulate}\, f_t)\,(\mathrm{tabulate}\, f_f)\\ &\quad \textbf{where}\quad f_t = \lambda s. f\,(\mathrm{true} :: s)\\ &\qquad\qquad\quad\;\; f_f = \lambda s. f\,(\mathrm{false} :: s)\end{aligned}$$

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Application of a tabulation:

$$\begin{array}{l} \mathsf{apply} : \mathbb{T}_A \to [\mathbb{B}] \to A \\ \mathsf{apply}\,(\mathsf{node}\, a\, t_1\, t_2)\,[] = a \\ \mathsf{apply}\,(\mathsf{node}\, a\, t_1\, t_2)\,(\mathsf{true} :: l) = \mathsf{apply}\, t_1\, l \\ \mathsf{apply}\,(\mathsf{node}\, a\, t_1\, t_2)\,(\mathsf{false} :: l) = \mathsf{apply}\, t_2\, l \end{array}$$

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A function $f : \mathbb{S}_A \to B$ can (?) be represented by a well-founded tree: [Ghani/Hancock/Pattinson:2006]

$$
\begin{array}{l}
\textbf{data } \mathbb{T}_{A,B} : \mathsf{Set} \\
\quad \mathsf{leaf} : B \to \mathbb{T}_{A,B} \\
\quad \mathsf{node} : (A \to \mathbb{T}_{A,B}) \to \mathbb{T}_{A,B}
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Is there an inverse transformation/tabulation?

$$\mathsf{tabulate} : (\mathbb{S}_A \rightarrow B) \rightarrow \mathbb{T}_{A,B}$$

Surely B must be a discrete/inductive type.

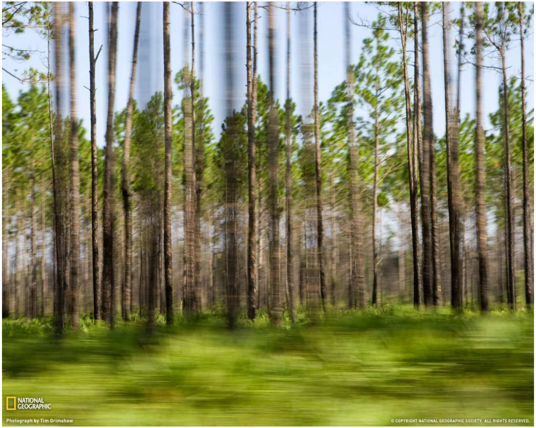
Tabulation Duality?

What is the relation between the two kinds of tabulations?

- Function on Inductive Types : Coinductive Tabulations.
- Function on CoInductive Types : Inductive Tabulations.

How to build a tabulation in the second case?
We need strong intentionality.

General Recursion



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$\quad \mathsf{found} : (s : \mathbb{S}_{\mathbb{N}})\mathsf{Dom}_{\mathsf{seek}}\,(0 :: s)$
$\quad \mathsf{notfound} : (n : \mathbb{N})(s : \mathbb{S}_{\mathbb{N}})\mathsf{Dom}_{\mathsf{seek}}\, s \to \mathsf{Dom}_{\mathsf{seek}}\,(\mathsf{S}\, n :: s)$

$\mathsf{seek} : (s : \mathbb{S}_{\mathbb{N}})\mathsf{Dom}_{\mathsf{seek}}\, s \to \mathbb{N}$
$\mathsf{seek}\,(0 :: s)\,(\mathsf{found}\, z) = 0$
$\mathsf{seek}\,(\mathsf{S}\, n :: s)\,(\mathsf{notfound}\, n\, s\, h) = \mathsf{S}\,(\mathsf{seek}\, s\, h)$

Computation by Prophecy

Coinductive version of the domain:
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$$
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\textbf{codata}\ \mathsf{Trace}_{\mathsf{seek}} : \mathbb{S}_{\mathbb{N}} \to \mathsf{Prop} \\
\quad \mathsf{found} : (s : \mathbb{S}_{\mathbb{N}})\mathsf{Trace}_{\mathsf{seek}}\,(0 :: s) \\
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\end{array}
$$

$$
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In all these representations:
How do we effectively compute the function?
Coinductive objects don't automatically unfold.
Domain predicate must be proved.

Computation by Judgement Rewriting

To compute $\mathsf{seek}\,(7 :: 2 :: 5 :: \cdots)$
Assume the domain predicate and type-check the result:

$$h : \mathsf{Dom}_{\mathsf{seek}}\,(7 :: 2 :: 5 :: \cdots) \vdash \mathsf{seek}\,(7 :: 2 :: 5 :: \cdots)\; h : \mathbb{N}$$

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Connection with Non-Standard Type Theory [Martin-Löf 1988]

Mixing Induction and Coinduction

Mixed Inductive/Coinductive Types

Data structures with both: [Danielsson/Altenkirch 2009]

- Constructors that must be well-founded;
- Constructors that can be iterated infinitely

Example, equality for the partiality monad:

$$
\begin{aligned}
&\textbf{codata}\ (\simeq) : B^{\nu} \to B^{\nu} \to \mathrm{Prop} \\
&\quad \mathrm{eqstep} : (x_1, x_2 : B^{\nu}) x_1 \simeq x_2 \to \lhd x_1 \simeq \lhd x_2 \\
&\quad \mathrm{eqval} : (b : B) \lceil b \rceil \simeq \lceil b \rceil \\
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$x \downarrow b$ inductive convergence relation

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The ∞ marks arguments that need not be well-founded

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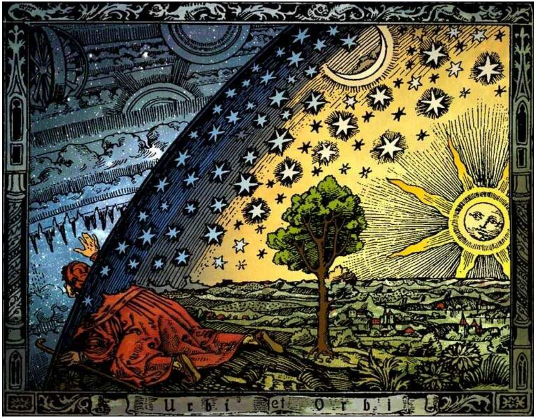
There are many other exiting topics.

MSFP



Mathematically Structured Functional Programming

Baltimore, 25 September 2010

http://cs.ioc.ee/msfp/msfp2010/

Deadline: 9 - 16 April