

Higher-order algebras and coalgebras from parameterized endofunctors

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Coalgebraic Methods in Computer Science 2010

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- Higher-order & parameterized endofunctors
- Initial and final suitability

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Basic Definitions OO O		
Higher-order & parameterized endofunctor		

Definition

A higher-order endofunctor is an endofunctor on a functor category.

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Basic Definitions OO O		
Higher-order & parameterized endofunctor		

Definition

A higher-order endofunctor is an endofunctor on a functor category.

Functor categories $[\mathcal{B}, \mathcal{C}]$, e.g.

- Category, $\mathcal{C} \cong [\mathbf{1}, \mathcal{C}]$.
- Arrow category, $\mathcal{C}^{\rightarrow} \cong [\mathbf{2}, \mathcal{C}].$

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- Monad category, Mon(C) (abusing terminology slightly)

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Basic Definitions ○●○ ○		
Higher-order & parameterized endofunctor		

Definition

A parameterized endofunctor is a bifunctor of the type $\mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$.

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Basic Definitions ○●○ ○		
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A parameterized endofunctor is a bifunctor of the type $\mathcal{B} \times \mathcal{C} \to \mathcal{C}$.

By currying, a parameterized endofunctor has type $\mathcal{B} \to \operatorname{End}(\mathcal{C}).$

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- There are more constrained notion of parameterized endofunctors, (e.g. structural actions (Blute-Cockett-Seely '97), parameterized monads (Uustalu '03, Atkey '06), etc.)

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- There are more constrained notion of parameterized endofunctors, (e.g. structural actions (Blute-Cockett-Seely '97), parameterized monads (Uustalu '03, Atkey '06), etc.)
- We use the unconstrained definition studied by Kurz and Pattinson '00.

Basic Definitions		
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Higher-order & parameterized endofunctors		

Parameterized endofunctors to higher-order endofunctors

Definition

For
$$F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$$
, let $\widehat{F}: [\mathcal{B}, \mathcal{C}] \to [\mathcal{B}, \mathcal{C}]$ be given by

$$\widehat{F}(X)(b) = F(b, Xb)$$

for $X: \mathcal{B} \to \mathcal{C}$ and $b \in \mathcal{B}$. \widehat{F} is the higher-order endofunctor generated by the parameterized endofunctor F.

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Basic Definitions ○ ○		
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Parameterized endofunctors to higher-order endofunctors

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for $X: \mathcal{B} \to \mathcal{C}$ and $b \in \mathcal{B}$. \widehat{F} is the higher-order endofunctor generated by the parameterized endofunctor F.

Goal

Characterize initial algebras and final coalgebras of these higher-order endofunctors in terms of lower-order properties.

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Basic Definitions	Applications	
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Initial and final suitability		

Suitable Parameterized Endofunctors

Definition

A parameterized endofunctor $F \colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ is

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Basic Definitions	Applications	
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Suitable Parameterized Endofunctors

Definition

A parameterized endofunctor $F \colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ is

• *initially suitable* if F(b, -) admits an initial algebra for any $b \in \mathcal{B}$.

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Basic Definitions ○○○ ●		
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Suitable Parameterized Endofunctors

Definition

A parameterized endofunctor $F \colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ is

- *initially suitable* if F(b, -) admits an initial algebra for any $b \in \mathcal{B}$.
- finally suitable if F(b, -) admits a final coalgebra for any $b \in \mathcal{B}$.

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Results	Applications	

An initially suitable parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a \mathcal{C} -endofunctor \mathcal{R}_F :

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Results	Applications	

An initially suitable parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a \mathcal{C} -endofunctor \mathcal{R}_F :

• $F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$ is the initial F(x, -)-algebra.

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	Results	Applications	
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An initially suitable parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a \mathcal{C} -endofunctor \mathcal{R}_F :

•
$$F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$$
 is the initial $F(x, -)$ -algebra.
• $\mathcal{R}_F f$ is induced by initiality.

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$$\begin{array}{c|c} F(x, \mathcal{R}_F x) & \xrightarrow{r_x} & \mathcal{R}_F x \\ F(x, \mathcal{R}_F f) & F(f, \mathcal{R}_F f) = F(-, \mathcal{R}_F -)f & & & & \\ F(x, \mathcal{R}_F y) & \xrightarrow{r_y} & F(y, \mathcal{R}_F y) & \xrightarrow{r_y} & \mathcal{R}_F y \end{array}$$

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•
$$\widehat{F}(R_F) = F(-, \mathcal{R}_F -) \xrightarrow{r} \mathcal{R}_F$$
 is a natural isomorphism.

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 is the initial $F(x, -)$ -algebra.

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$$\widehat{F}(R_F) = F(-, \mathcal{R}_F -) \stackrel{r}{\Rightarrow} \mathcal{R}_F$$
 is a natural isomorphism.
• $\widehat{F}\mathcal{R}_F \stackrel{r}{\Rightarrow} \mathcal{R}_F$ is an \widehat{F} -algebra!

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Results	Applications	

An initially suitable parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a \mathcal{C} -endofunctor \mathcal{R}_F :

•
$$F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$$
 is the initial $F(x, -)$ -algebra.

- $\mathcal{R}_F f$ is induced by initiality.
- $\widehat{F}(R_F) = F(-, \mathcal{R}_F -) \xrightarrow{r} \mathcal{R}_F$ is a natural isomorphism. • $\widehat{F}\mathcal{R}_F \xrightarrow{r} \mathcal{R}_F$ is an \widehat{F} -algebra!
- Let $S_F \stackrel{s}{\Rightarrow} \widehat{F}S_F$ be the \widehat{F} -coalgebra induced by finally suitable parameterized endofunctors.

Results	

Theorem (J.Kim '09)

Let \widehat{F} be a $[\mathcal{B}, \mathcal{C}]$ -endofunctor generated by a parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$. The following are equivalent:

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Results	

Theorem (J.Kim '09)

Let \widehat{F} be a $[\mathcal{B}, \mathcal{C}]$ -endofunctor generated by a parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$. The following are equivalent:

- **1** *F* is initially (resp. finally) suitable.
- **2** \widehat{F} admits an initial algebra (resp. final coalgebra).
- If F is initially suitable, $\widehat{F}\mathcal{R}_F \stackrel{r}{\Rightarrow} \mathcal{R}_F$ is the initial \widehat{F} -algebra.

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- If F is finally suitable, $S_F \stackrel{s}{\Rightarrow} \widehat{F}S_F$ is the final \widehat{F} -coalgebra.
- The result can be specialized to "parameterized monads"
 F: B → Mon(C). A monad structure can be imposed on the higher-order [B,C]-endofunctor F̂.

Example

Let $G_1 \stackrel{\theta}{\Longrightarrow} G_0$ be a natural transformation between two C-endofunctors. Let $D: \mathbf{2} \times C \to C$ be given by

$$D(i, x) = G_i(x)$$
 and $D(!, x) = \theta_x$

for
$$i \in \mathbf{2}$$
, $x \in \mathcal{C}$. Recall $\mathbf{2} = \underbrace{0 \xleftarrow{!} 1}_{(\mathrm{id}_1)}$.

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• D is initially suitable if G_0 and G_1 admit initial algebras.

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- D is initially suitable if G_0 and G_1 admit initial algebras.
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- The initiality version of the theorem generalizes a result by Chuang and Lin '06, proved for arrow categories.

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Applications

Conclusions

An Example of a Parameterized Endofunctor II

Example

Let H be a C-endofunctor. Let $E: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be given by

$$E(a,x) = a + Hx$$

for $a, x \in \mathcal{C}$.

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Applications

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- \mathcal{R}_E is the free monad generated by H.
- S_E is the completely iterative monad generated by H.

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- \mathcal{R}_E is the free monad generated by H.
- S_E is the completely iterative monad generated by H.
- E is finally suitable $\iff E$ is iteratable
- The finality version of the theorem generalizes a result by Aczel, Adámek, Milius, Velebil '03, proved for iteratable endofunctors.

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Example

Let A, B be nonempty sets. Let $F : (Set^{op} \times Set) \times Set \rightarrow Set$ be given by $F(\langle A, B \rangle, C) = (B \times C)^A$.

Higher-order algebras and coalgebras from parameterized endofunctors



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Example

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•
$$\mathcal{S}_F \langle A, B \rangle = \Gamma_{A,B} = \{f \colon A^{\omega} \to B^{\omega} : f \text{ causal}\}.$$
 Let $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$ be given by

$$\gamma_{A,B}(f)(a) = \langle \mathsf{hd} \circ f \circ c_a, \mathsf{tl} \circ f \circ c_a \rangle$$

for $f \in \Gamma_{A,B}$, $a \in A$, $c_a(\sigma) = a : \sigma$. (Rutten)

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• The first coordinate $hd \circ f \circ c_a$ is a constant in B since f is causal.

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	Applications	

map as a higher-order coalgebra morphism I Recall the final \hat{F} -coalgebra

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Basic Definitions 000 0		Applications	
map as a higher-or Recall the final \widehat{F} \widehat{F} is a [Set ^{op}	order coalge -coalgebra × Set, Set]-end	ebra morphism l dofunctor.	

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Higher-order algebras and coalgebras from parameterized endofunctors

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		Applications	
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		Applications	
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- Let $\mathcal{S}_F = \Gamma \colon \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set} \to \mathsf{Set}$ be given by $\Gamma(A, B) = \Gamma_{A, B}$.
- $\Gamma \stackrel{\gamma}{\Longrightarrow} \widehat{F}(\Gamma) = F(-, \Gamma-)$ is the final \widehat{F} -coalgebra.

Basic Definitions 000 0		Applications	
map as a higher-or Recall the final \widehat{F} -co \widehat{F} is a [Set ^{op} × \blacksquare Let $S_F = \Gamma$: Se $\blacksquare \Gamma \xrightarrow{\gamma} \widehat{F}(\Gamma) = I$ Define another \widehat{F} -co	der coalgebra mo palgebra Set, Set]-endofuncto $et^{op} \times Set \rightarrow Set$ be $F(-, \Gamma-)$ is the final palgebra	prphism ${\sf I}$ r. given by $\Gamma(A,B)=\Gamma_A$ \widehat{F} -coalgebra.	,В.

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Higher-order algebras and coalgebras from parameterized endofunctors

Jiho Kim

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map as a higher-order coalgebra morphism l
Recall the final \widehat{F} -coalgebra
• \widehat{F} is a [Set ^{op} × Set, Set]-endofunctor.
• Let $\mathcal{S}_F = \Gamma \colon Set^{op} \times Set \to Set$ be given by $\Gamma(A, B) = \Gamma_{A, B}$.
• $\Gamma \stackrel{\gamma}{\Rightarrow} \widehat{F}(\Gamma) = F(-, \Gamma-)$ is the final \widehat{F} -coalgebra.
Define another \widehat{F} -coalgebra
• Let Hom : $\operatorname{Set}^{\operatorname{op}} \times \operatorname{Set} \to \operatorname{Set}$ be the usual hom-bifunctor.

Applications

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		Applications	
map as a high	er-order coalge	bra morphism I	
Recall the fina	al \widehat{F} -coalgebra		
$\blacksquare \widehat{F}$ is a [S	$et^{op} imes Set, Set] ext{-enc}$	lofunctor.	
• Let \mathcal{S}_F =	$: \Gamma: Set^{op} \times Set \to$	Set be given by $\Gamma(A, B)$	$P() = \Gamma_{A,B}.$
$ \Gamma \xrightarrow{\gamma} \widehat{F}($	$\Gamma) = F(-, \Gamma-)$ is t	the final \widehat{F} -coalgebra.	
Define anothe	r \widehat{F} -coalgebra		
Let Hom	$: Set^op \times Set \to Set$	et be the usual hom-bifu	nctor.
Let Hom	$\stackrel{e}{\Rightarrow} \widehat{F} \operatorname{Hom} = F(-$	$(-, \operatorname{Hom} -)$ be a \widehat{F} -coalge	ebra where
the comp	onents		
	$\operatorname{Hom}(A,B) \xrightarrow{e_{\langle A,A}}$	$\xrightarrow{B} (B \times \operatorname{Hom}(A, B))^A$	

Applications

is given by

$$e_{\langle A,B\rangle}(f)(a) = \langle f(a),f\rangle.$$

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	Applications	

By finality of γ , there is a higher-order coalgebra morphism $\operatorname{Hom} \xrightarrow{m} \Gamma$:

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	Applications	

By finality of γ , there is a higher-order coalgebra morphism $\operatorname{Hom} \xrightarrow{m} \Gamma$:

• Hom
$$(A, B) \xrightarrow{e_{\langle A, B \rangle}} (B \times \operatorname{Hom}(A, B))^A$$

 $\downarrow^{(B \times m_{A,B})^A}$
 $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$

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	Applications	

By finality of γ , there is a higher-order coalgebra morphism $\operatorname{Hom} \stackrel{m}{\Longrightarrow} \Gamma$:



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	Applications	

By finality of γ , there is a higher-order coalgebra morphism $\operatorname{Hom} \stackrel{m}{\Longrightarrow} \Gamma$:



Overview

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	Conclusions
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Overview

1 Higher-order endofunctors & parameterized endofunctors

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	Conclusions

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- 1 Higher-order endofunctors & parameterized endofunctors
- Characterization of initial algebras and final coalgebras for higher-order endofunctors generated by parameterized endofunctors.

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Overview

- 1 Higher-order endofunctors & parameterized endofunctors
- Characterization of initial algebras and final coalgebras for higher-order endofunctors generated by parameterized endofunctors.
- **3** Generalization of known results.

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	Conclusions

Overview

- 1 Higher-order endofunctors & parameterized endofunctors
- Characterization of initial algebras and final coalgebras for higher-order endofunctors generated by parameterized endofunctors.
- **3** Generalization of known results.
- 4 Derivation of map as a higher-order coalgebra morphism.

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 Algebraic and coalgebraic properties of higher-order endofunctors should be studied.

	Conclusions

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Conclusions

- Algebraic and coalgebraic properties of higher-order endofunctors should be studied.
- 2 Particulars of other "constrained" functor categories should be studied.

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