Regular Languages of Trees and Probability

Matteo Mio
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Outline of the Talk

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- Introduction to SAT problem for ordinary temporal logics.
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- Connection with theory of Regular languages of trees.
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- Connection with theory of *Regular languages of trees*.
- Motivation: the SAT problem for probabilistic logics.
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Results: probability-related facts about regular languages:
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1. *Measurability,*
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Results: probability-related facts about regular languages:

1. Measurability,
2. Closure Properties,
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- Motivation: the SAT problem for probabilistic logics.

Results: probability-related facts about regular languages:

1. Measurability,
2. Closure Properties,
3. How to compute their probability.
**Temporal Logics**

**Models** = Directed Graphs with predicates:

\[ S \rightarrow \mathcal{P}(\text{Prop}) \times \mathcal{P}(S) \]

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Models = Directed Graphs with predicates:
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\[
\begin{array}{c}
p \\ q \\ r \\ s
\end{array}
\]

\[ P = \{ p, q, s \} \]

Logics = $\mu$-calculus, CTL, CTL*, ...
Temporal Logics

**Models** = Directed Graphs with predicates:
\[ S \rightarrow \mathcal{P}(\text{Prop}) \times \mathcal{P}(S) \]

\[
\begin{tikzpicture}
  \node (p) at (0,0) {p};
  \node (q) at (1,-1) {q};
  \node (r) at (2,0) {r};
  \node (s) at (1,-2) {s};

  \path[->]
  (p) edge (q)
  (q) edge (r)
  (r) edge (s)
  (s) edge (p);
\end{tikzpicture}
\]

\[ \text{Prop} = \{P\}, \ P = \{p, q, s\} \]

**Logics** = \(\mu\)-calculus, CTL, CTL\(^*\), ...

- There exists an infinite path of states satisfying \(P\):
  \[ \nu X. \lozenge (P \land X) \]
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Given a finite model $M$ and a formula $\phi$,

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- Decidable.
Two Main Problems

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Given a formula $\phi$,

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* Decidable.
* Finite Model Property.
Monadic Second Order Logic

\[
\text{MSO} = \text{First order logic} + \text{Monadic 2nd order quantification}
\]

\[
\phi \lor \psi \mid \neg \phi \mid \exists x.\phi(x) \mid \exists X.\phi(X) \mid x \in X
\]
Monadic Second Order Logic

MSO = First order logic + Monadic 2nd order quantification

\[ \phi \lor \psi \mid \neg \phi \mid \exists x. \phi(x) \mid \exists X. \phi(X) \mid x \in X \]

MSO is interpreted over a fixed model, the FULL BINARY TREE.

- Relational structure \( \langle \{L, R\}^*, Succ_L, Succ_R \rangle \),
- where \( Succ_L(x) = x.L \) and \( Succ_R(x) = x.R \)
Example 1:

\[ \forall x. \exists y. (y = \text{Succ}_L(x)) \]
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Example 2:
\[ \exists x. \forall y. (x \neq \text{Succ}_L(y) \land x \neq \text{Succ}_R(y)) \]
Example 1:
\[ \forall x. \exists y. (y = Succ_L(x)) \quad \text{TRUE} \]

Example 2:
\[ \exists x. \forall y. (x \neq Succ_L(y) \land x \neq Succ_R(y)) \quad \text{TRUE} \quad x = \epsilon \]
Example 1:
\[ \forall x. \exists y. \left( y = \text{Succ}_L(x) \right) \quad \text{TRUE} \]

Example 2:
\[ \exists x. \forall y. \left( x \neq \text{Succ}_L(y) \land x \neq \text{Succ}_R(y) \right) \quad \text{TRUE} \quad x = \epsilon \]

Example 3:
\[ \exists X. \left( \epsilon \in X \land \forall x \in X \rightarrow \text{Succ}_L(x) \in X \right) \]
Example 1:
\[ \forall x. \exists y. (y = \text{Succ}_L(x)) \quad \text{TRUE} \]

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\[ \exists x. \forall y. (x \neq \text{Succ}_L(y) \land x \neq \text{Succ}_R(y)) \quad \text{TRUE} \quad x = \epsilon \]

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Theorem (M. Rabin ’69): The MSO theory of the full binary tree is decidable.
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1. Take a (CTL, \(\mu\)-calculus, \ldots) formula \(F(P_1, \ldots, P_n)\)
   - E.g., \(\mu\)-calculus formula \(\nu X.\Box(X \land P)\)

2. translate it to a MSO formula \(\phi(P_1, \ldots, P_n)\),
   - \(\exists X.(\text{"\(X\) is an infinite branch" } \land \forall x. x \in X \rightarrow x \in P)\)

3. check if \(\exists P_1 \ldots \exists P_n.\phi(P_1, \ldots, P_n)\) is valid.

**Caveat:** This only checks satisfiability of \(F(P_1, \ldots, P_n)\) by a model having a binary-tree structure.
How to solve the SAT problem for temporal logics:

1. Take a (CTL, $\mu$-calculus, ...) formula $F(P_1, \ldots, P_n)$
   - E.g., $\mu$-calculus formula $\nu X. \lozenge (X \land P)$
2. translate it to a MSO formula $\phi(P_1, \ldots, P_n)$,
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3. check if $\exists P_1 \ldots \exists P_n. \phi(P_1, \ldots, P_n)$ is valid.

Caveat: This only checks satisfiability of $F(P_1, \ldots, P_n)$ by a model having a binary-tree structure.
   - Interpret arbitrary trees by binary tree with "dummy states"
\[ M \models F(P_1, \ldots, P_n) \iff \llbracket M \rrbracket \models \phi(P_1, \ldots, P_n, D) \]
A predicate $P$ over the domain $\{L, R\}^*$ is a function:

$$P : \{L, R\}^* \rightarrow \{0, 1\}$$
Regular Languages of Trees

A predicate $P$ over the domain $\{L, R\}^*$ is a function:

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**Terminology:** the *space* of $\{0, 1\}$-labeled trees,

$$P : \{L, R\}^* \rightarrow \{0, 1\} \iff t \in T_{0,1}$$
A predicate $P$ over the domain $\{L, R\}^*$ is a function:

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**Terminology:** the space of $\{0, 1\}$-labeled trees,

$$P : \{L, R\}^* \rightarrow \{0, 1\} \iff t \in T_{0,1}$$

**Definition:** A set $L \subseteq T_{0,1}$ is regular if:

$$L = \{ t \mid \phi(t) \text{ holds} \}$$

for some MSO formula $\phi(X)$. 
Definition (extended): A set $L \subseteq \mathcal{T}_{0,1} \times \mathcal{T}_{0,1}$ is regular if:

$$L = \{ \langle t_1, t_2 \rangle \mid \phi(t_1, t_2) \text{ holds} \}$$

for some MSO formula $\phi(X, Y)$.
Definition (extended): A set $L \subseteq \mathcal{T}_{0,1} \times \mathcal{T}_{0,1}$ is regular if:

$$L = \{ \langle t_1, t_2 \rangle \mid \phi(t_1, t_2) \text{ holds} \}$$

for some MSO formula $\phi(X, Y)$.

Definition (final): A set $L \subseteq (\mathcal{T}_{0,1})^n$ is regular if:

$$L = \{ \langle t_1, t_2, \ldots, t_n \rangle \mid \phi(t_1, t_2, \ldots, t_n) \text{ holds} \}$$

for some MSO formula $\phi(X_1, \ldots, X_n)$. 
Logical connectives as set-theoretical operations:

- $L = \phi \implies L^c = \neg \phi$
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- $L = \phi \quad \Rightarrow \quad L^c = \neg \phi$

- $L_1 = \phi_1 \text{ and } L_2 = \phi_2 \quad \Rightarrow \quad L_1 \cup L_2 = \phi_1 \lor \phi_2$
Logical connectives as set-theoretical operations:

- \( L = \phi \implies L^c = \neg \phi \)
- \( L_1 = \phi_1 \) and \( L_2 = \phi_2 \implies L_1 \cup L_2 = \phi_1 \lor \phi_2 \)
- \( L = \phi(X, Y) \implies \exists X. \phi(X, Y) \) is the projection
The set $\mathcal{T}_{0,1}$ is a metric (Polish) space.
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**Diagram:**

- $\Delta^0_1$ arrows to $\Sigma^0_1$ and $\Pi^0_1$
- $\Delta^0_2$ arrows to $\Sigma^0_2$ and $\Pi^0_2$
- $\ldots$ arrows to $\Delta^0_\alpha$, $\Sigma^0_\alpha$, and $\Pi^0_\alpha$
- $\Delta^0_{\alpha+1}$ arrows to $\ldots$

**Borel** $= \Delta^1_1$

- $\Delta^1_2$
- $\Delta^1_3$
- $\ldots$

- $\Pi^1_1$
- $\Pi^1_2$
- $\Pi^1_3$

**Notation:**

- $\Delta^0_{\alpha+1}$
- $\Sigma^0_{\alpha}$
- $\Pi^0_{\alpha}$
- $\Delta^1_{\alpha+1}$
- $\Sigma^1_{\alpha}$
- $\Pi^1_{\alpha}$
The set $\mathcal{T}_{0,1}$ is a metric (Polish) space.

$\Delta_0 \to \Sigma_1 \to \Delta_1 \to \Pi_1 \to \Delta_2 \to \Pi_2 \to \ldots \Delta_\alpha \to \Pi_\alpha \to \Delta_{\alpha+1} \to \ldots$

$\Sigma_0 \to \ldots \to \Sigma_\alpha \to \ldots$

Borel $= \Delta_1^1$
$\text{Borel} = \Delta^1_1$
**Paper:** Arnold and Niwinski, *Continuous Separation of Game Languages*, in Fundamenta Informaticae 2007.

Game Languages $W_{0,k}$, for $k > 0$

- For every regular $L \subseteq \mathcal{T}_{0,1}$ there exists $k$ such that $L \leq W_{0,k}$.  

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Game Languages $W_{0,k}$, for $k > 0$

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**Theorem:** $W_{0,1} \not\leq W_{0,2} \not\leq W_{0,3} \ldots$
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For every regular $L \subseteq \mathcal{T}_{0,1}$ there exists $k$ such that $L \leq W_{0,k}$.

**Theorem:** $W_{0,1} \preceq W_{0,2} \preceq W_{0,3} \ldots$
Points discussed so far

- SAT problem for temporal logics ($\mu$-calculus, CTL, CTL*,...).
- MSO as a general solution for the SAT problem.
- Regular Languages as Boolean algebra of sets in the Polish space $T_{0,1}$. 
Models = Markov Chains with Predicates:
\[ S \rightarrow \mathcal{P}(\text{Prop}) \times \mathcal{D}(S) \]

\[ Prop = \{ P \}, \ P = \{ p, q, s \} \]
**Probabilistic Temporal Logics**

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\[
\begin{array}{c}
p \quad 1 \\
q \quad 1 \\
\quad 1/2 \\
s \quad 1/2
\end{array}
\]

\( p \to q \quad 1 \)
\( q \to r \quad 1 \)
\( q \to s \quad 1/2 \)
\( r \to p \quad 1 \)

\( \text{Prop} = \{P\}, \ P = \{p, q, s\} \)

**Logics** = probabilistic CTL (pCTL), probabilistic \( \mu \)-calculus, etc.
Probabilistic Temporal Logics

Models = Markov Chains with Predicates:
\[ S \to \mathcal{P}(\text{Prop}) \times \mathcal{D}(S) \]

\[
\begin{array}{c}
p \quad 1 \\
q \quad 1 \quad \frac{1}{2} \quad r \\
s \quad \frac{1}{2} \quad \frac{1}{2} \\
\end{array}
\]

\[ \text{Prop} = \{P\}, \ P = \{p, q, s\} \]

Logics = probabilistic CTL (pCTL), probabilistic \(\mu\)-calculus, etc.

- The probability of generating an infinite path of states satisfying \(P\) is \(> 0.85\): \(\mathbb{P}_{>0.85}(G \ P)\)
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- Open Problem !!!
Failure of Finite Model Property

**Property:** “The probability of producing an infinite path of states satisfying $\neg a \land \psi$ is positive”

$$\mathbb{P}_{>0}(G(\neg a \land \psi))$$
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where $\psi$ says: “the probability of reaching a state satisfying $a$ is positive”: $\psi = \mathbb{P}_{>0}(\circ a)$
Failure of Finite Model Property

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P_{>0}(G (\neg a \land \psi))
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where \( \psi \) says: “the probability of reaching a state satisfying \( a \) is positive”: \( \psi = P_{>0}(\circ a) \)
Finite-SAT Problem: Given a formula $\phi$,

$$\exists M. ( M \models \phi ) \land M \text{ is finite?}$$
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General SAT Problem: Given a formula $\phi$, 
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- If a model $M$ exists, is it finitely presentable?
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General SAT Problem: Given a formula $\phi$,

$$\exists M. (M \models \phi)$$

- If a model $M$ exists, is it finitely presentable?
- Are the probabilities appearing in $M$ rational, algebraic, computable?
**Theorem** (LICS 2008, Brázdil, Forejt, Kretínský, Kucera)

Both problems are decidable for **qualitative** pCTL.

- Only constraints $= 0$, $\geq 0$, $< 1$ and $= 1$
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Natural Questions:

1. Can we extend it to quantitative pCTL?
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**Natural Questions:**

1. Can we extend it to **quantitative** pCTL?
2. Can we extend it to qualitative fragments of more expressive logics

   - pCTL*, pECTL*, probabilistic $\mu$-calculus, etc?
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This talk: can we use knowledge about MSO to solve these problems?
MSO logic is interpreted over the Full Binary Tree
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Idea: Interpret it as a binary Markov chain with coin-flip transitions.
Step 1: Replace $n$-ary transitions by binary transitions.
**Step 1:** Replace $n$-ary transitions by binary transitions.

**Step 2:** Replace generic probabilistic transitions by coin-flip transitions.
Extension of MSO

\[ MSO ::= \phi \lor \psi \mid \neg \phi \mid \forall x.\phi(x) \mid \forall X.\phi(X) \mid x \in X \]

**New** “for almost all” quantifier: \( \forall^{=1} X.\phi(X) \)
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- \( \phi(X) \) holds on a *random* predicate \( X \) with probability 1
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**New** "for almost all" quantifier: \( \forall^=1 X.\phi(X) \)

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- \( \mu(\{t \mid \phi(t) \text{ holds}\}) = 1 \)

where \( \mu \) is the coin-flipping probability measure on the space \( \mathcal{T}_{0,1} \)
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- Example: \( \mu(\{ t \mid \epsilon \text{ is labeled by } 0 \}) = \frac{1}{2} \)
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- Example: \( \mu\left(\{t \mid \epsilon \text{ and } L \text{ are labeled by 0}\}\right) = \frac{1}{4} \)
Fact: MSO+∀=1 can encode qualitative pCTL*, pECTL*, probabilistic μ-calculus, . . .
Fact: MSO+$\forall^=1$ can encode qualitative pCTL*, pECTL*, probabilistic $\mu$-calculus, ... 

\[ \forall^=1 X.\phi(X) \iff \mu\left(\{t \mid \phi(t) \text{ holds}\}\right) = 1 \]
Fact: MSO+$\forall=^1$ can encode qualitative pCTL*, pECTL*, probabilistic $\mu$-calculus, ... \\

$$\forall=^1 X.\phi(X) \Leftrightarrow \mu(\{ t \mid \phi(t) \text{ holds} \}) = 1$$

Question: is the regular set $\phi(t)$ measurable?
Question: Are regular sets $L \subseteq \mathcal{T}_{0,1}$ measurable?
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Example: regular set $L \subseteq \mathcal{T}_{0,1}$

$$\{ t \mid \text{has } \geq |\mathbb{N}_1| \text{ branches with infinitely many 1’s} \}$$
Measurability of Regular Sets

**Question:** Are regular sets $L \subseteq T_{0,1}$ measurable?

"Temporary" Solution (PhD thesis, 2012): $\text{ZFC} + \text{MA}_{\aleph_1}$:
Kolmogorov’s $R$-sets
Goal (1928): Find a large $\sigma$-algebra of safe sets.
Kolmogorov’s $\mathcal{R}$-sets

**Goal (1928):** Find a large $\sigma$-algebra of *safe* sets.

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**Idea:** define operator (transform) $\mathcal{R}$ acting on operations.
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Idea: define operator (transform) $\mathcal{R}$ acting on operations.

- $\mathcal{R}(\bigcup_n) = \mathcal{A}$. 
Kolmogorov’s $\mathcal{R}$-sets

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- $\mathcal{R}(\bigcup_n) = \mathcal{A}$.
- $\mathcal{R}(\mathcal{A})$ a new and more expressive operation on sets.
Kolmogorov’s $\mathcal{R}$-sets

**Goal (1928):** Find a large $\sigma$-algebra of safe sets.

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- $R(\bigcup_n) = A.$
- $R(A)$ a new and more expressive operation on sets.
- $RR(A)$ ...

**Kolmogorov’s $\sigma$-algebra of $R$-sets:** $\sigma(\text{Open}, \{R^n\}_n, \neg)$
Theorem (1928): Every $\mathcal{R}$-set is measurable.
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**Theorem (1928):** Every $\mathcal{R}$-set is measurable.

**Goal:** We want to show that all regular sets are $\mathcal{R}$-sets.
$\mathcal{R}$-sets can be classified by their order

**Def:** $\text{rank}(X) = n \iff X = \mathcal{R}^n(U_0, \ldots , U_n, \ldots)$
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**Theorem** (MFCS 2014, Gogacz, Michalewski, Mio, Skrzypczak)

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- Every Regular language in an \( \mathcal{R} \)-set.
- The game language \( W_{0,n} \) is complete for the \( n \)-level of the hierarchy of \( \mathcal{R} \)-sets.
\( R \)-sets can be classified by their order

**Def:** \( \text{rank}(X) = n \iff X = R^n(U_0, \ldots, U_n, \ldots) \)

**Theorem** (MFCS 2014, Gogacz, Michalewski, Mio, Skrzypczak)

- Every Regular language in an \( R \)-set.
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- The class of \( R \)-set is precisely the class of sets constructible with parity games (with infinite arenas).
R-sets can be classified by their order

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Back to MSO + $\forall^=1$

\[
\phi \lor \psi \mid \neg \phi \mid \forall x.\phi(x) \mid \forall X.\phi(X) \mid x \in X \mid \forall^=1 X.\phi(X)
\]
In terms of sets, the quantifier $\forall^=1 X.\phi(X)$ takes "large sections":

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Quantifier $\forall^=1 X. \phi(X)$ first studied by H. Friedman in 1979.
Other “for almost all” quantifiers include:

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- For uncountably many: $\forall \geq N_1 X. \phi(X)$ (Mostowski)
- For comeager many: $\forall^* X. \phi(X)$ (Friedman)

**Theorem:** (Barany, Kaiser, Rabinovich, CSL 2009)

$$\text{MSO} = \text{MSO} + \forall \geq N_1$$
Question: Does $\text{MSO} = \text{MSO} + \forall^1$?
Question: Does MSO = MSO + $\forall^1$ ?

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*Proof Sketch:* Quantifier elimination exploiting well-known Banach–Mazur game interpretation of (co)meagerness.
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**Input:** formula \( \forall^* X. \phi(X, Y) \)

**Output:** automaton accepting the language of \( \forall^* X. \phi(X, Y) \)
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- (Hjorth, Khoussainov, Montalban, LICS 2008)
- It also follows from the theory of \( \mathcal{R} \)-sets.
A surprising result of Ludwig Staiger (CSL 1996)

**Theorem:** A regular set of $\omega$-words $L \subseteq \Sigma^\omega$ is comeager if and only if it has coin-flipping measure 1.
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**Theorem:** A regular set of $\omega$-words $L \subseteq \Sigma^\omega$ is comeager if and only if it has coin-flipping measure 1.

It follows from this fact that:

**Corollary:** The finite-SAT problem for qualitative pCTL, pCTL*, pECTL* and probabilistic $\mu$-calculus is decidable.
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**Theorem:** A regular set of $\omega$-words $L \subseteq \Sigma^\omega$ is comeager if and only if it has coin-flipping measure 1.

It follows from this fact that:

**Corollary:** The finite-SAT problem for qualitative pCTL, pCTL*, pECTL* and probabilistic $\mu$-calculus is decidable.

- Complements the result of Brázdil, Forejt, Kretínský, Kucera (LICS 2008).
- Proof method is applicable to many variants of pCTL.
What about MSO + $\forall=^1$?

Can we say something interesting about MSO + $\forall=^1$?
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- Can define non-regular sets. ☐
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Simplest formula involving the $\forall^{=1}$ quantifier:

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with $\phi(X)$ without occurrences of $\forall^{=1}$ quantifiers.
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i.e., $\mu(\{t \mid \phi(t)\}) = 1$?
Problem: Compute the probability $\mu(L)$ of a regular set $L$. 
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Algorithm (Michalewski, Mio)

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- works only on $L$ recognized by game–automata.

**General Problem**: open.

- Computing the Rabin–Mostowski index is also open.
Conclusions

• Some open problems
  ▸ SAT for probabilistic logics,
  ▸ Identifying decidable fragments of \( MSO + \forall = 1 \),
  ▸ Computing the probability \( \mu(L) \) of regular sets \( L \).

• Interplay with descriptive set theory:
  ▸ Question about measurability of regular sets.
    ▹ led to connection with Kolmogorov’s \( R \)-sets.
  ▸ Classical notion of large section (comeager).
    ▹ led to \( MSO = MSO + \forall^* \) and decidability of finite-SAT problem.
“Using” the Algorithm

Game Languages $\mathcal{W}_{0,k}$ are definable by game automata.
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Fact 1: $\mu(W_{0,k}) = 1$ if $k$ is even and $\mu(W_{0,k}) = 0$ if $k$ is odd.
Fact 2: There are regular sets $L$ with irrational probability $\mu(L)$. 

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Example: $L_1 \subseteq \mathcal{T}_{\{a,b,c\}}$

\[ \mu(L_1) = \frac{1}{2} \]
“Using” the Algorithm

Fact 2: There are regular sets $L$ with irrational probability $\mu(L)$.

- All game-automata definable $L$ have algebraic probability.

Example: $L_2 \subseteq \mathcal{T}_{\{a,b,c\}}$

\[ \mu(L_2) = \frac{1}{4}(3 - \sqrt{7}) \approx 0.088 \]
“Using” the Algorithm

Example: \( L_3 \subseteq T_{\{a,b,c\}} \)

\[
\mu(L_3) = \frac{1}{4}(3 - \sqrt{1 + 3\sqrt{7}}) \approx 0.0026
\]

\[
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Fact 3: Let \( L_\infty = \bigcap_n L_n \).
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- Staiger’s property for trees is false.
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