Regular Languages of Trees and Probability

Matteo Mio
CNRS – ENS Lyon

Outline of the Talk

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Results: probability-related facts about regular languages:

1. *Measurability*,
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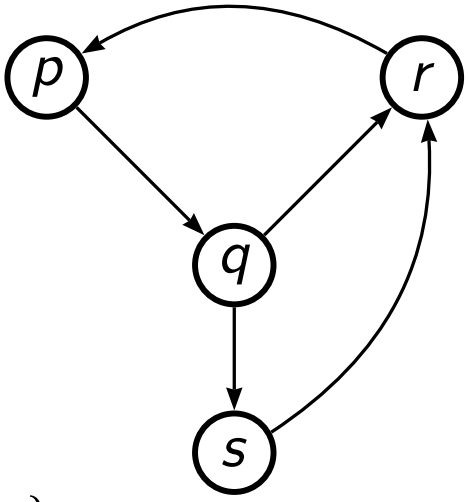
3. *How to compute their probability*.

Temporal Logics

Models = Directed Graphs with predicates:
$S \to \mathcal{P}(\text{Prop}) \times \mathcal{P}(S)$



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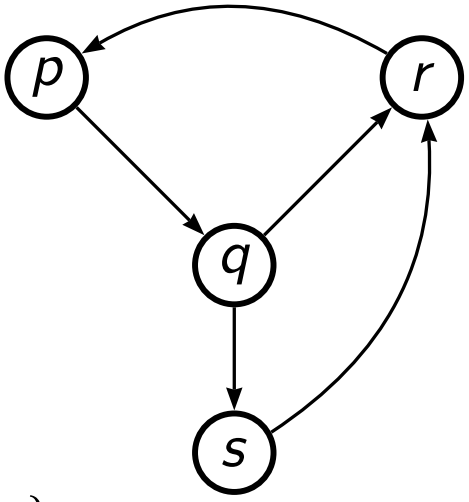

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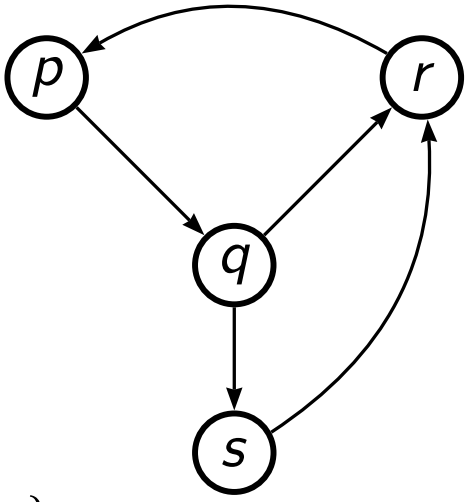

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- There exists an infinite path of states satisfying P:
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Two Main Problems

Problem 1: **Model Checking**

Given a finite model M and a formula ϕ,

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- Finite Model Property.

Monadic Second Order Logic

MSO = First order logic + Monadic 2nd order quantification

$$\phi \vee \psi \mid \neg\phi \mid \exists x.\phi(x) \;\mid\; \exists X.\phi(X) \mid x \in X$$

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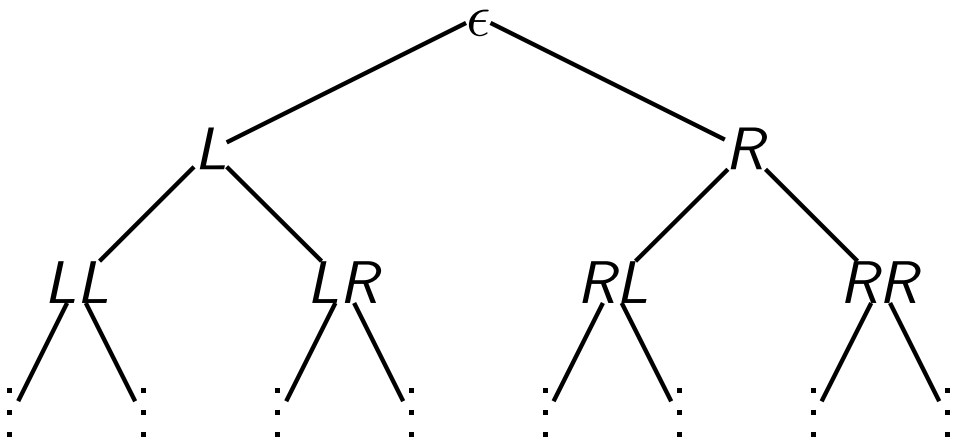
MSO is interpreted over a <u>fixed</u> model, the FULL BINARY TREE.

- Relational structure $\langle \{L, R\}^*, Succ_L, Succ_R \rangle$,
- where $Succ_L(x) = x.L$ and $Succ_R(x) = x.R$



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Theorem (M. Rabin '69): The MSO theory of the full binary tree is decidable.

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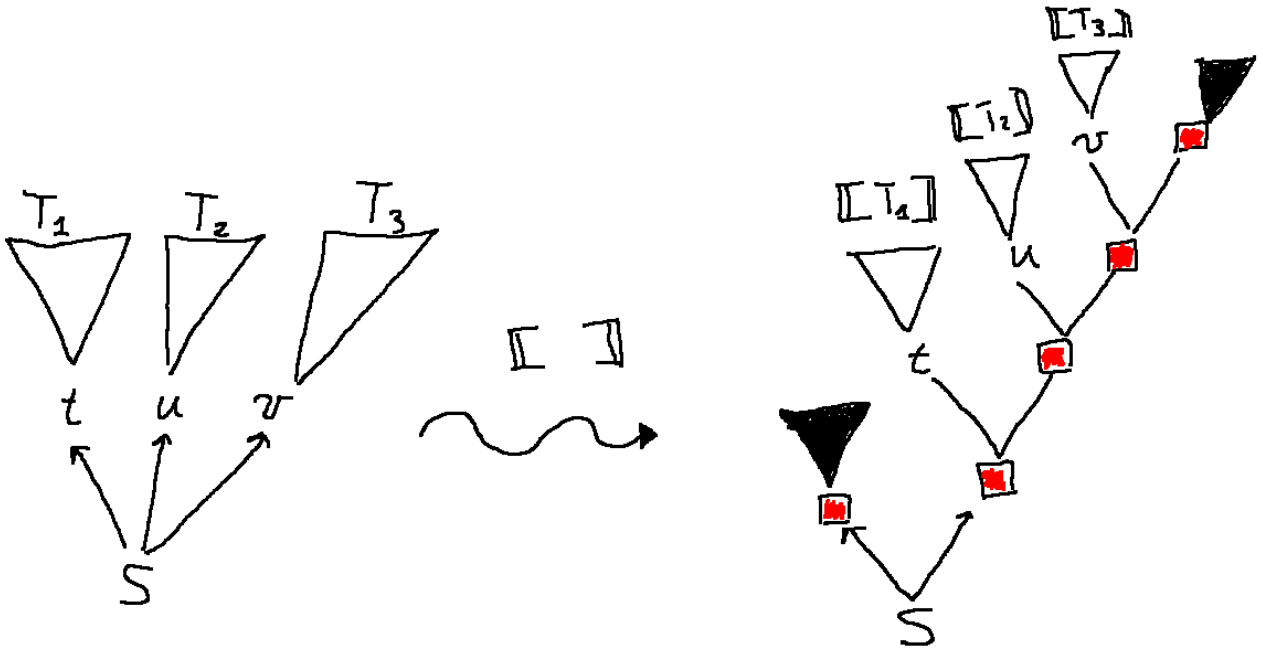
- Interpret arbitrary trees by binary tree with "dummy states"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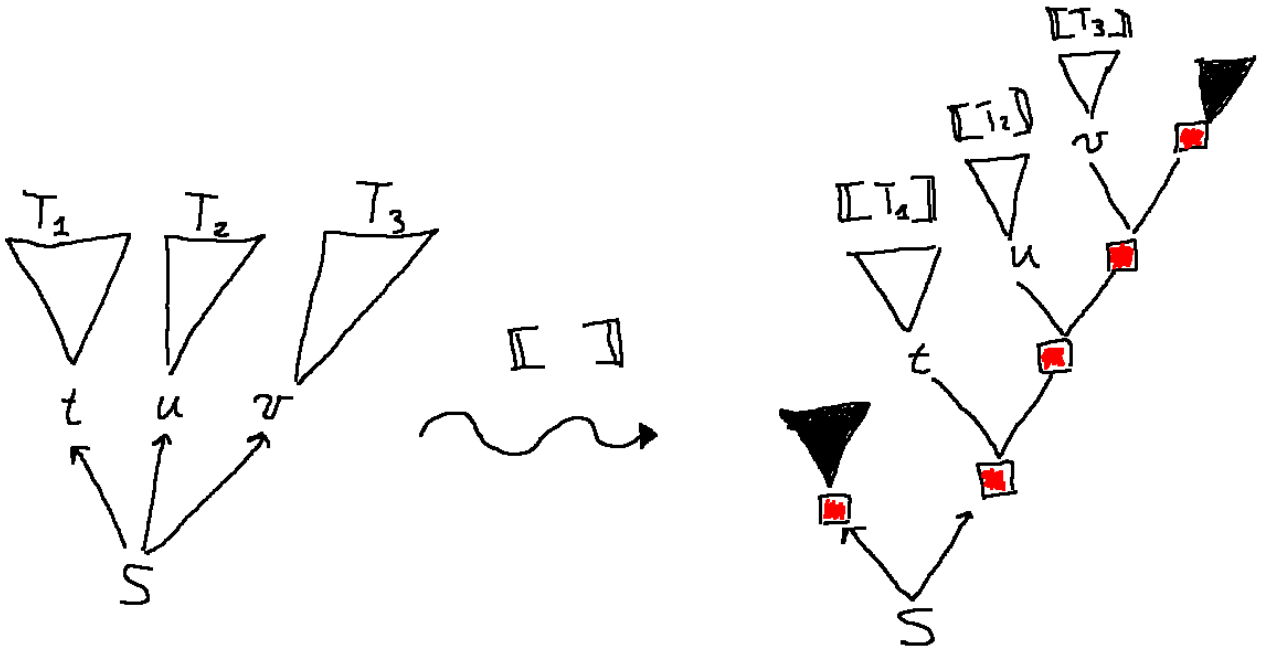

$$M \models F(P_1, \ldots, P_n) \quad \Leftrightarrow \quad [\![M]\!] \models \phi(P_1, \ldots, P_n, D)$$

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Definition: A set $L \subseteq \mathcal{T}_{0,1}$ is <u>regular</u> if:

$$L = \big\{ t \mid \phi(t) \quad \text{holds} \big\}$$

for some MSO formula $\phi(X)$.

Definition (extended): A set $L \subseteq \mathcal{T}_{0,1} \times \mathcal{T}_{0,1}$ is <u>regular</u> if:

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Definition (final): A set $L \subseteq (\mathcal{T}_{0,1})^n$ is <u>regular</u> if:

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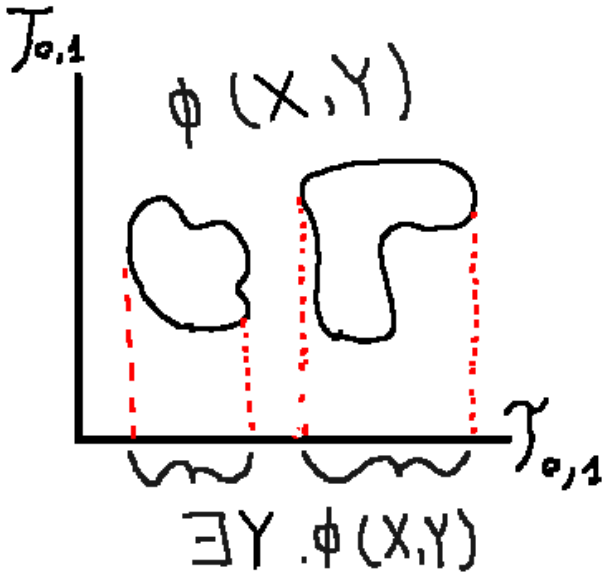
- $L = \phi(X, Y) \quad \Rightarrow \quad \exists X.\phi(X, Y)$ is the projection



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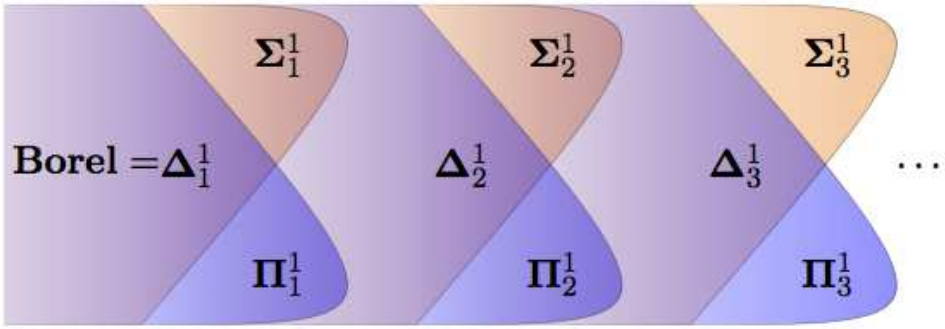

Topology and Descriptive Set Theory

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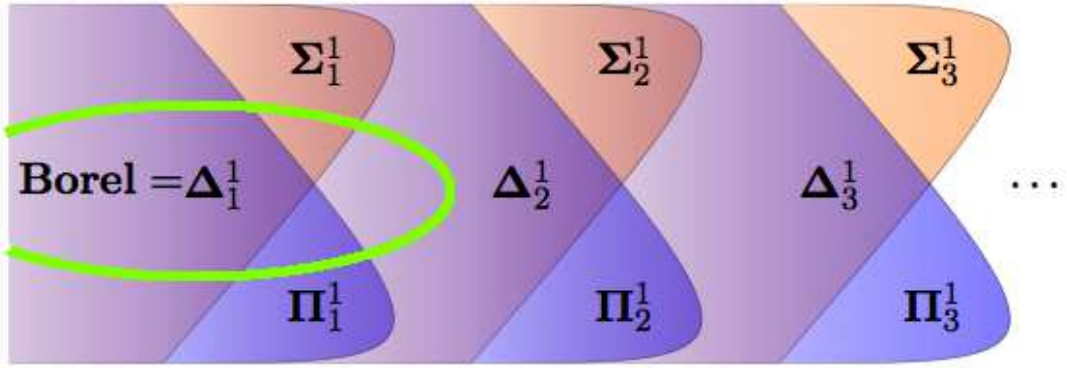

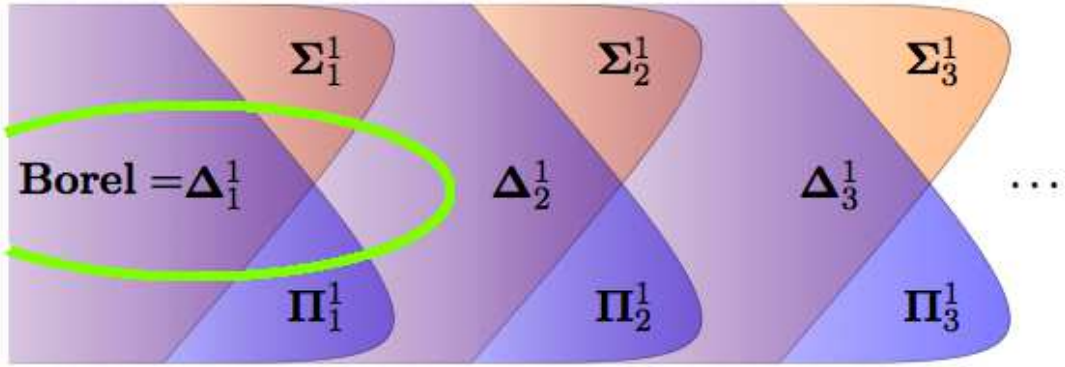
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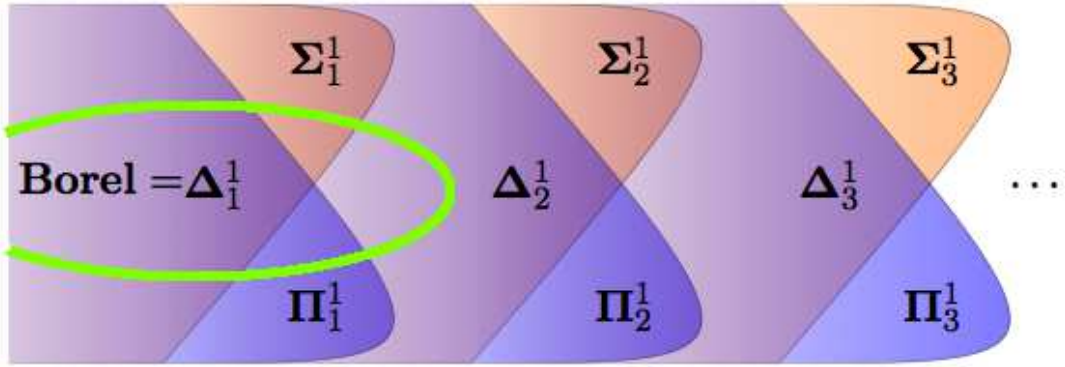

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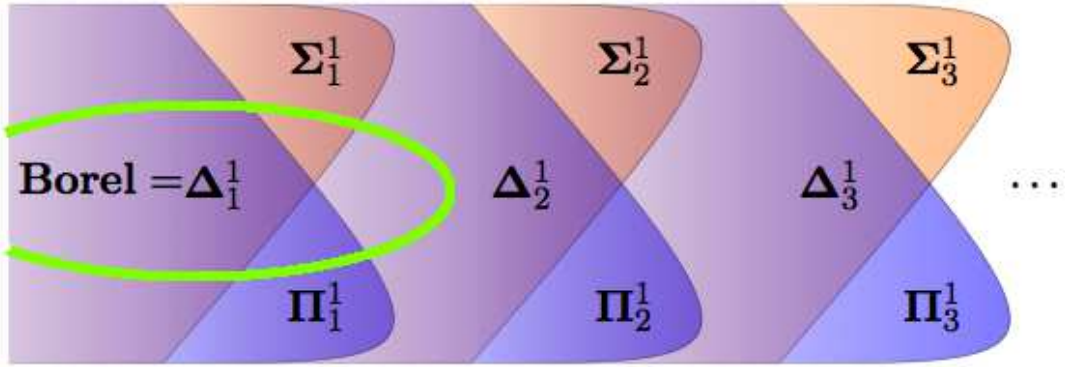

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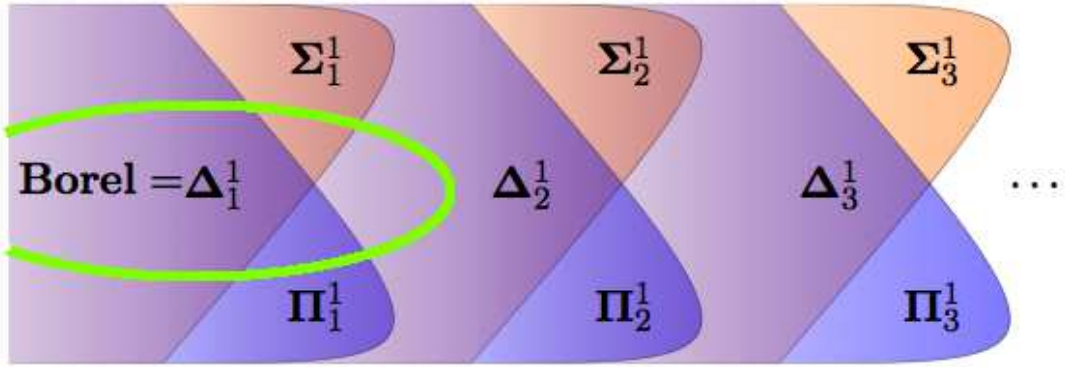

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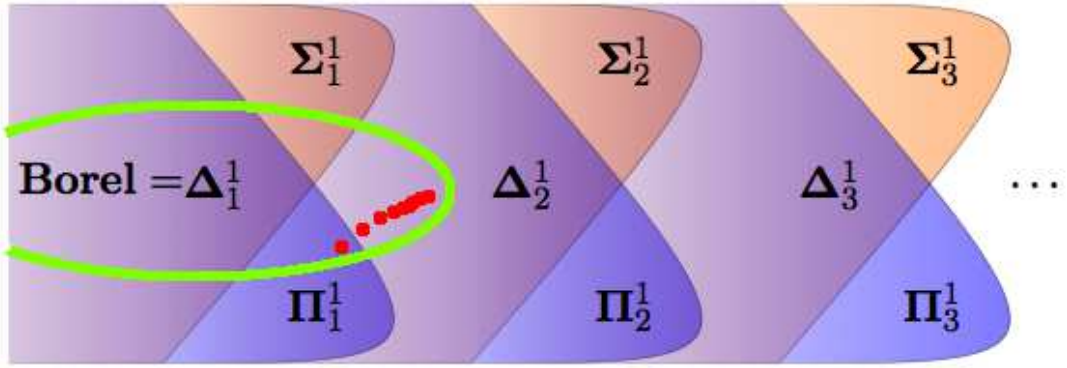

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Points discussed so far

- SAT problem for temporal logics (μ-calculus, CTL, CTL*,. . .).
- MSO as a general solution for the SAT problem.
- Regular Languages as Boolean algebra of sets in the Polish space $\mathcal{T}_{0,1}$.

Probabilistic Temporal Logics

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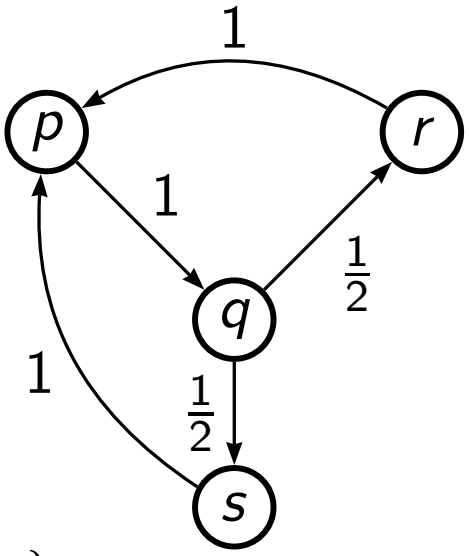

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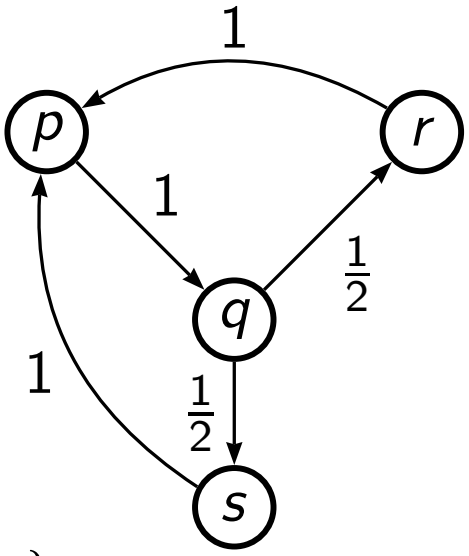

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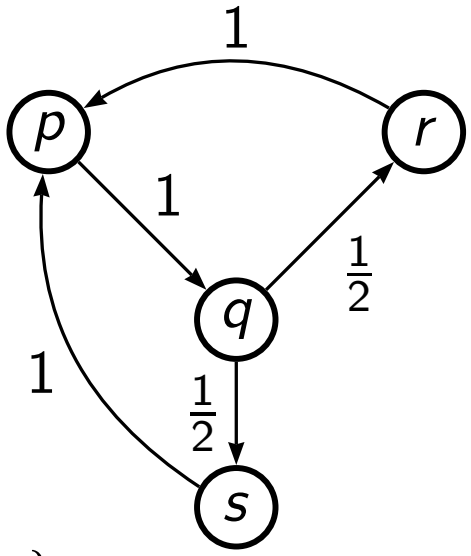

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- The probability of generating an infinite path of states satisfying P is > 0.85: $\mathbb{P}_{>0.85}\big(\mathsf{G}\ P\big)$

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- Open Problem **!!!**

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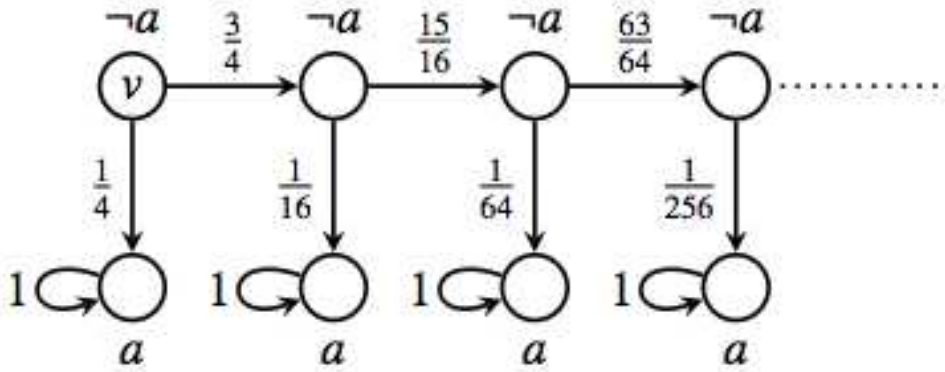

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General SAT Problem: Given a formula ϕ,

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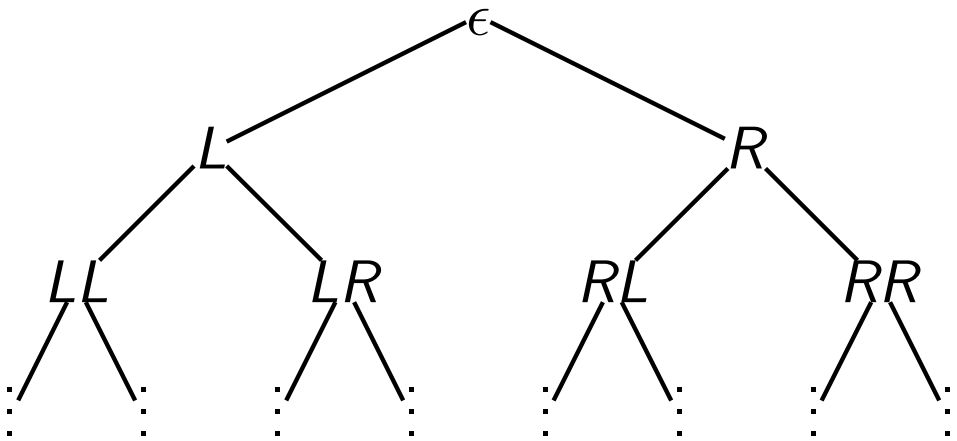
This talk: can we use knowledge about MSO to solve these problems?

MSO logic is interpreted over the Full Binary Tree



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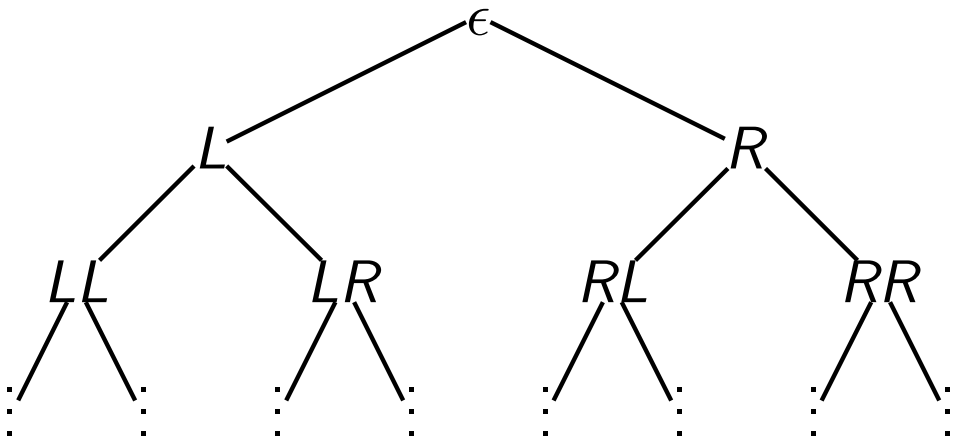

Idea: Interpret it as a binary Markov chain with coin-flip transitions.

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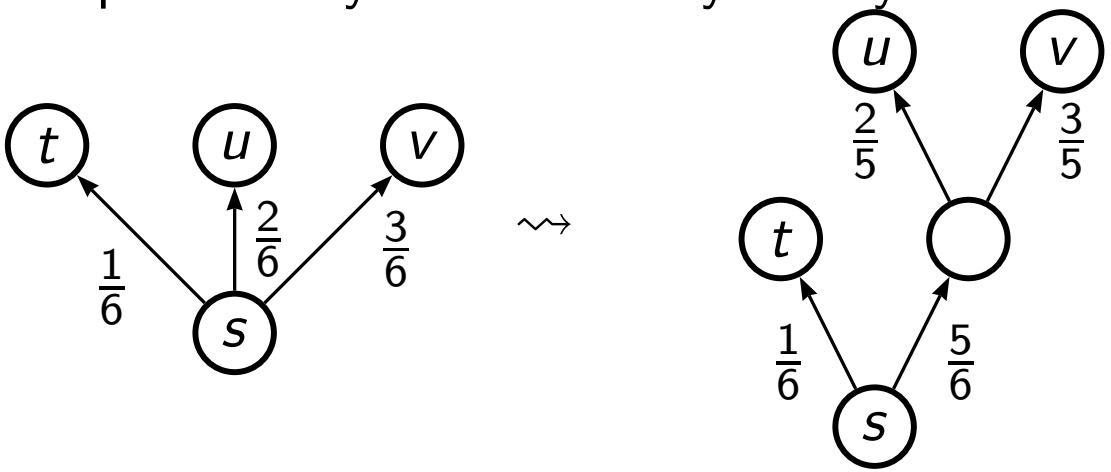

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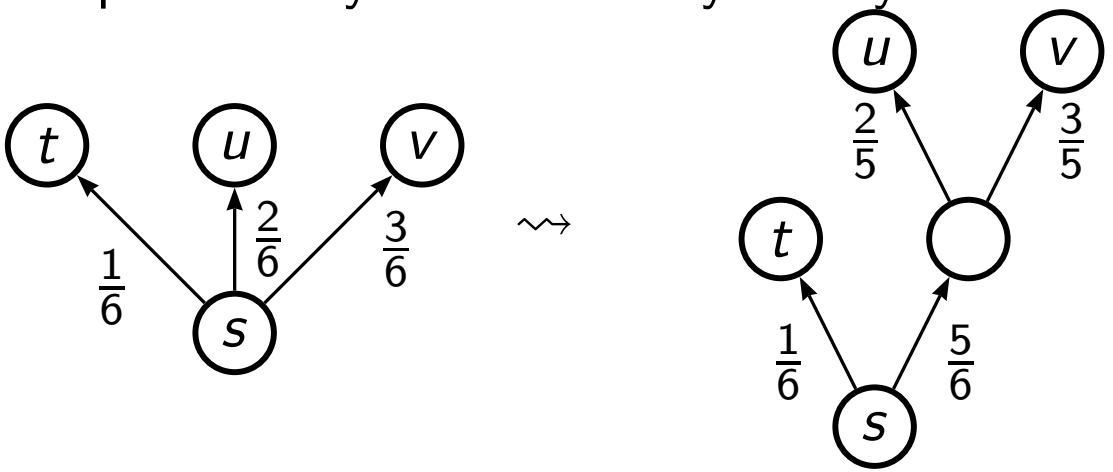
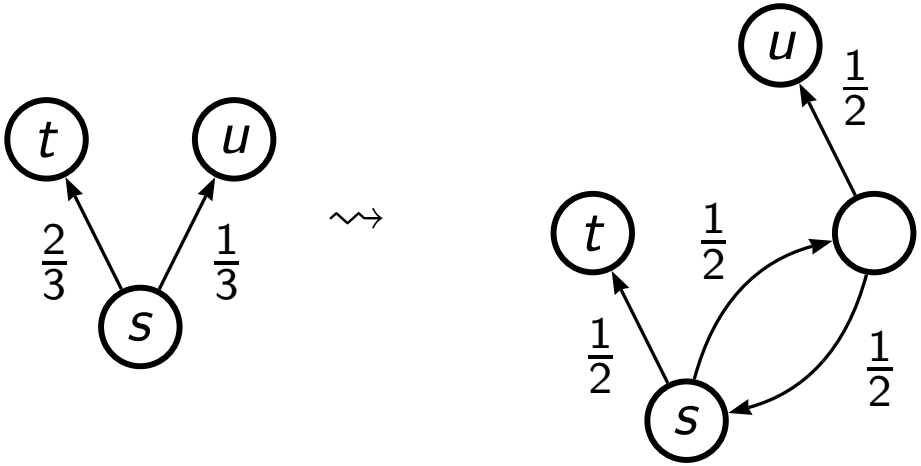


Step 2: Replace generic probabilistic transitions by coin-flip transitions.



Extension of MSO

$$MSO ::= \phi \vee \psi \mid \neg\phi \mid \forall x.\phi(x) \;\mid\; \forall X.\phi(X) \mid x \in X$$

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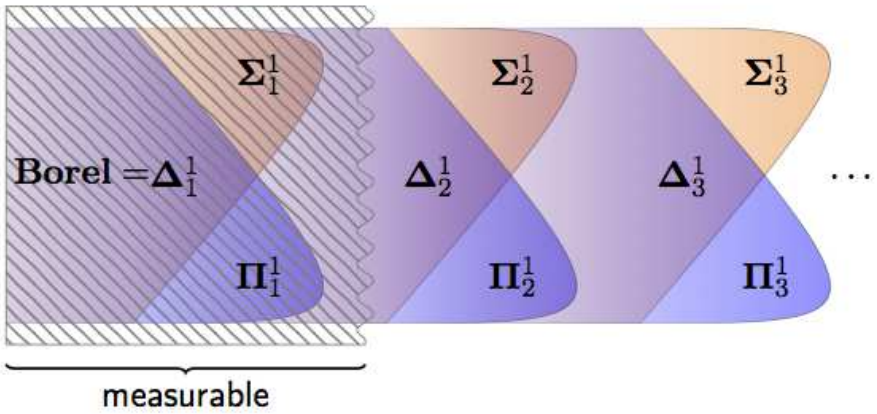

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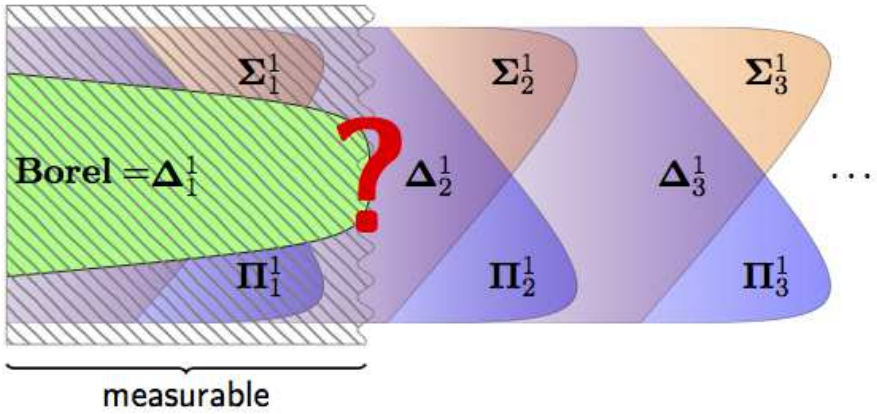

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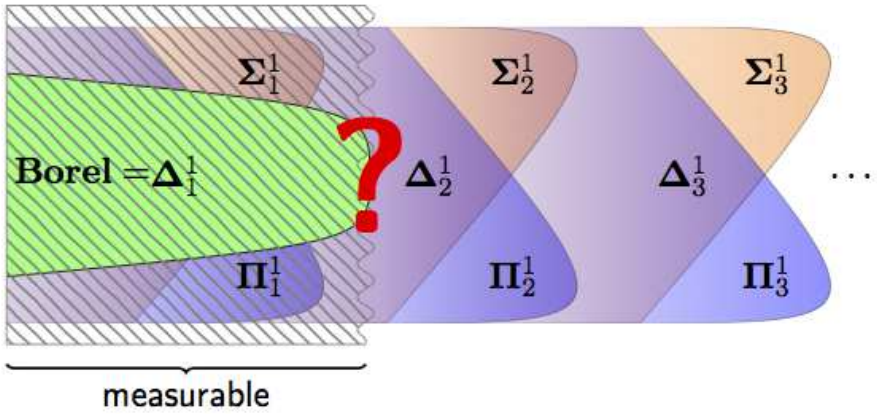

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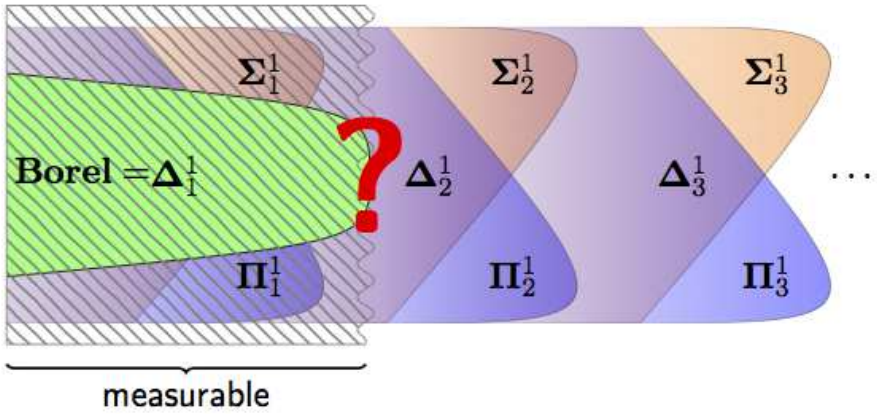
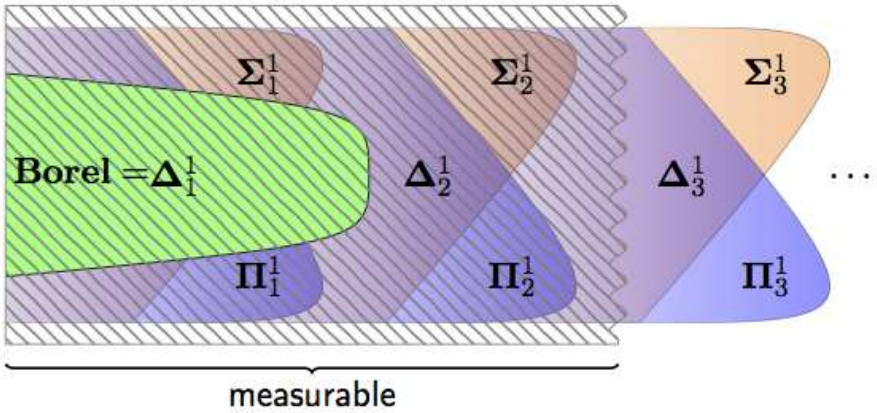


"Temporary" Solution (PhD thesis, 2012): ZFC + $\mathsf{MA}_{\aleph_1}$:



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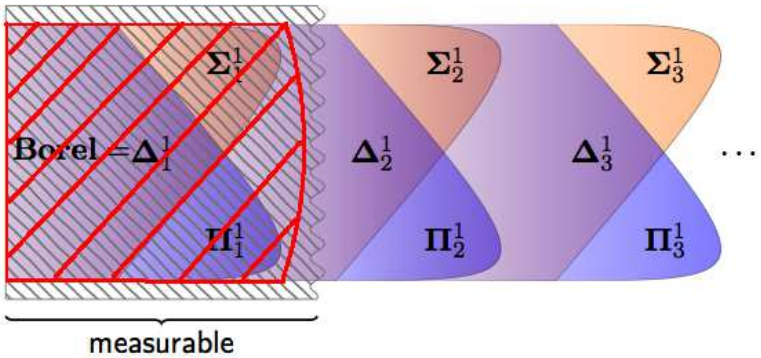

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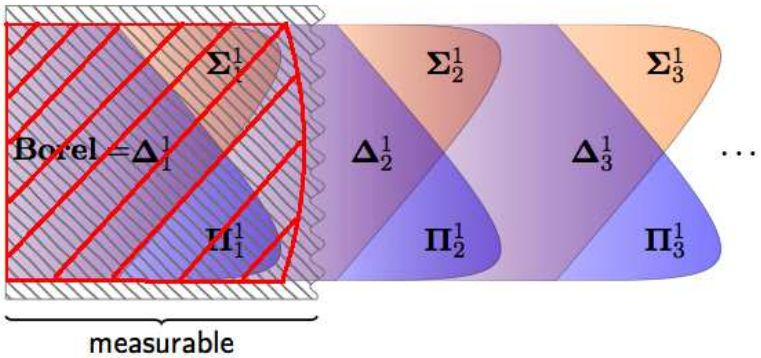
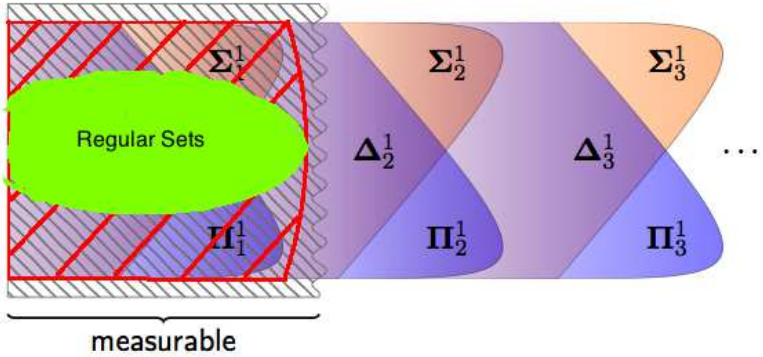


Goal: We want to show that all regular sets are $\mathcal{R}$-sets.



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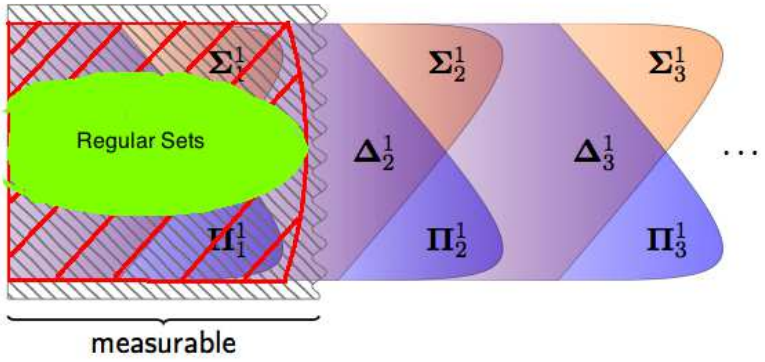

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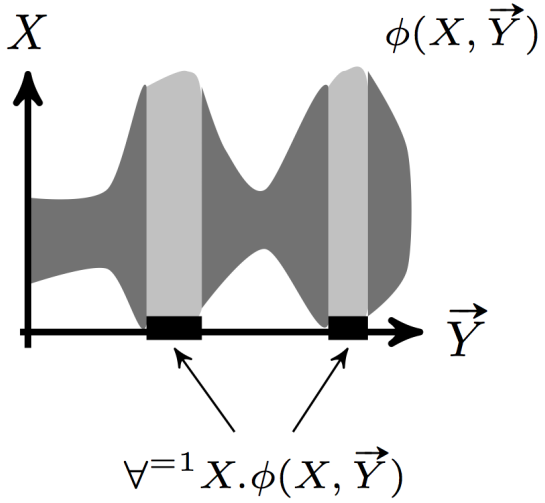

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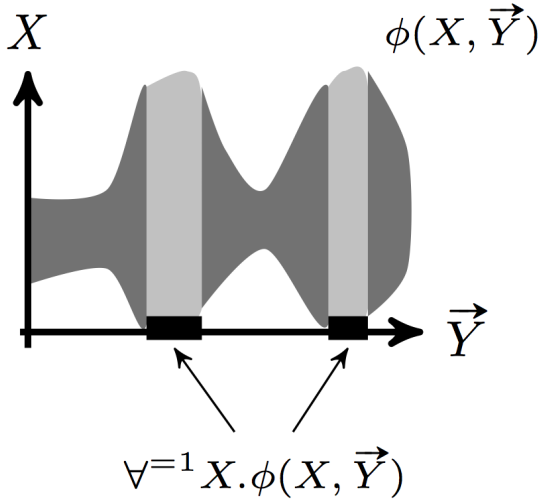
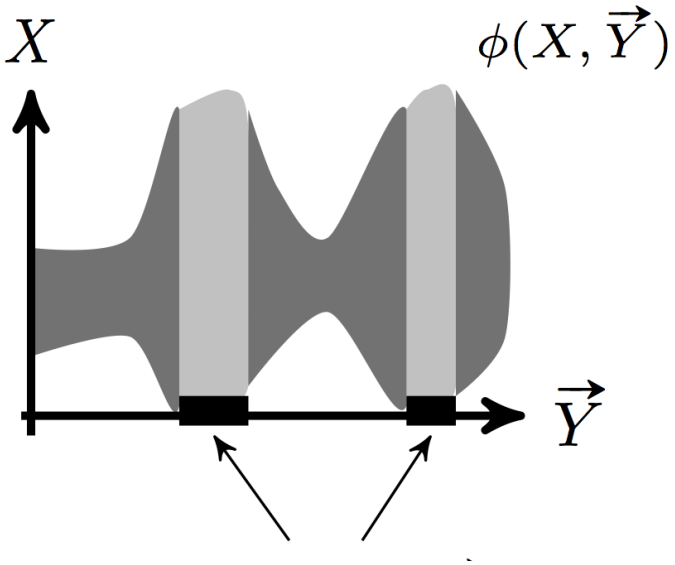


Quantifier $\forall^{=1} X.\phi(X)$ first studied by H. Friedman in 1979.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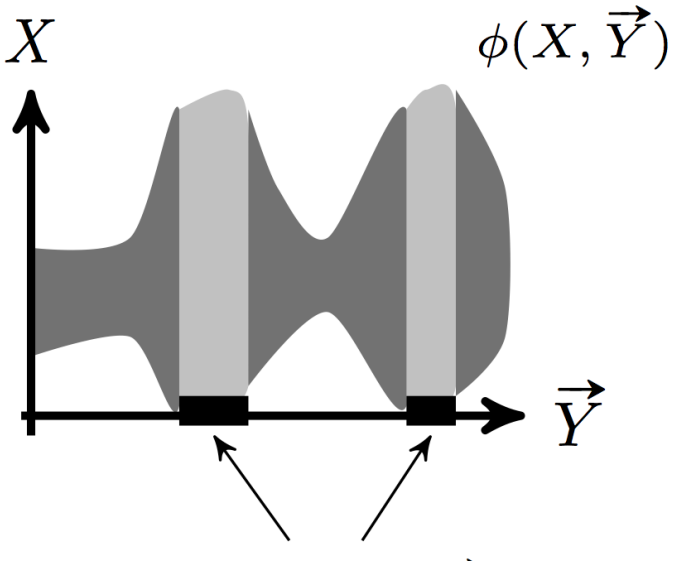

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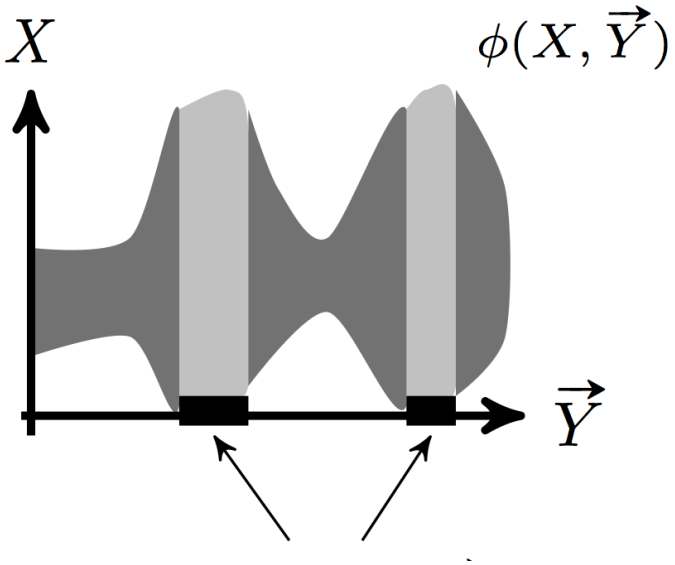

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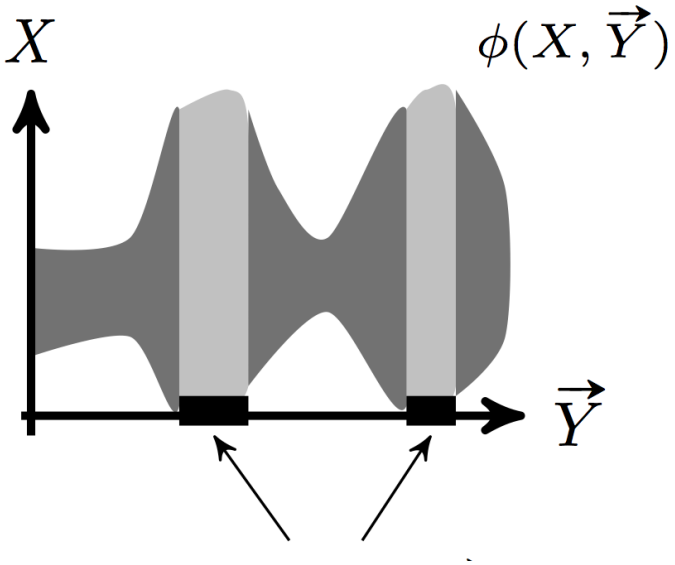

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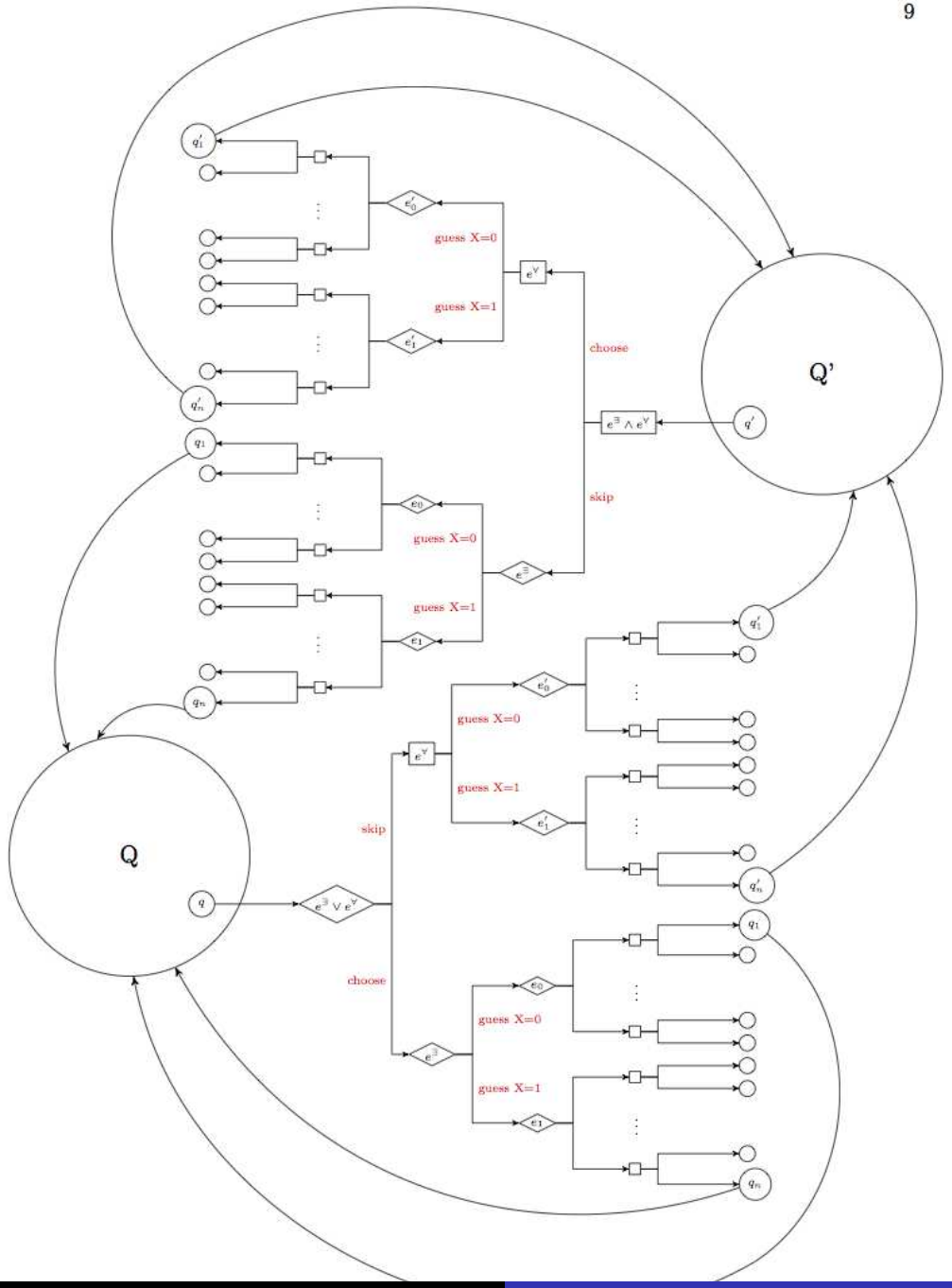
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- Complements the result of Brázdil, Forejt, Kretínský, Kucera (LICS 2008).
- Proof method is applicable to many variants of pCTL.

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$$\text{i.e.,} \qquad \mu\big(\{t \mid \phi(t)\}\big) = 1?$$

Computing the Probability of Regular Sets

Problem: Compute the probability $\mu(L)$ of a regular set L.

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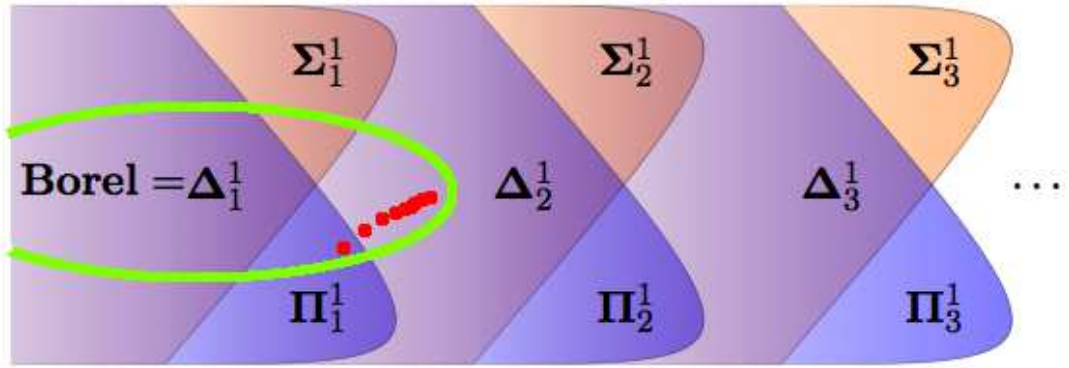
"Using" the Algorithm

Game Languages $W_{0,k}$ are definable by game automata.



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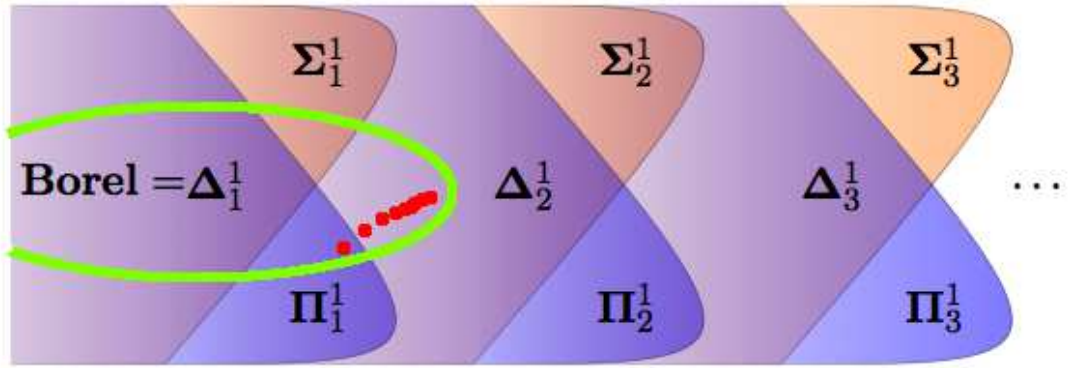

Fact 1: $\mu(W_{0,k}) = 1$ if k is even and $\mu(W_{0,k}) = 0$ if k is odd.

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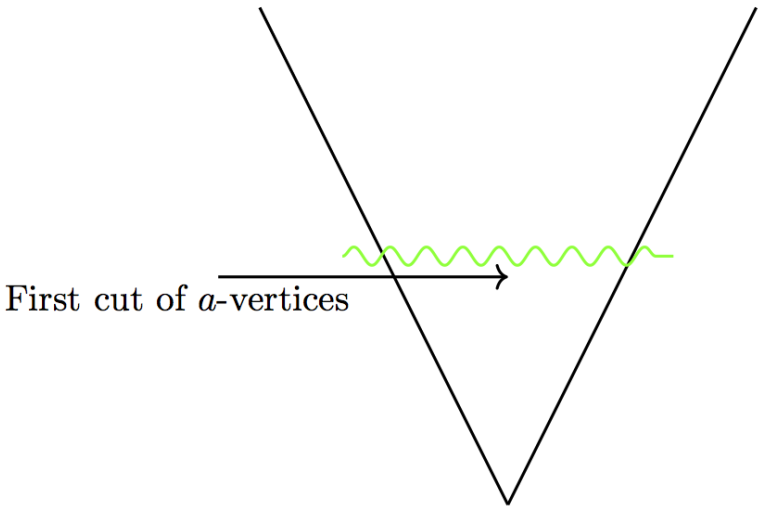

$\mu(L_1) = \frac{1}{2}$

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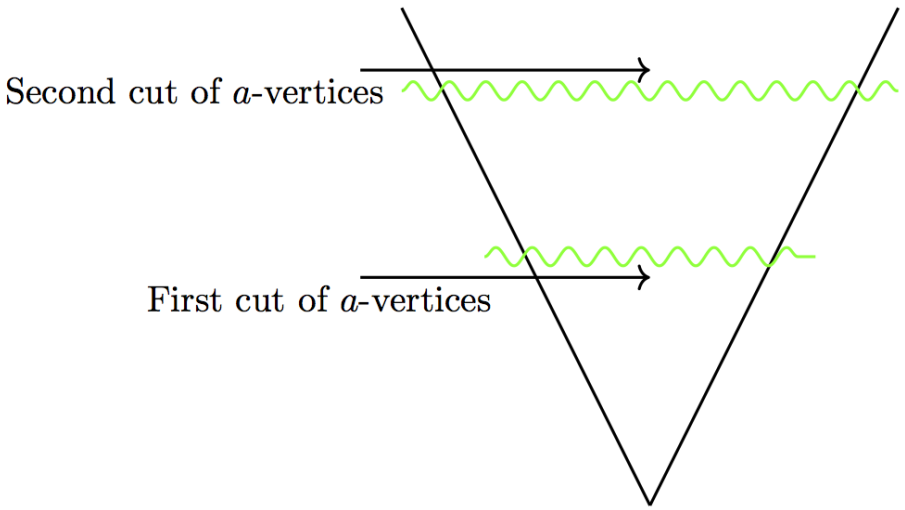

$\mu(L_2) = \frac{1}{4}(3 - \sqrt{7}) \approx 0.088$

"Using" the Algorithm

Example: $L_3 \subseteq \mathcal{T}_{\{a,b,c\}}$

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$\mu(L_1) = \frac{1}{2} \qquad \mu(L_2) \approx 0.088 \qquad \mu(L_3) \approx 0.0026\ldots$

Fact 3: Let $L_\infty = \bigcap_n L_n$.

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- Staiger's property for trees is <u>false</u>.

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