Revisiting the Institutional Approach to Herbrand's Theorem

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Abstract

More than a decade has passed since Herbrand's theorem was first generalized to arbitrary institutions, enabling in this way the development of the logic-programming paradigm over formalisms beyond the conventional framework of relational first-order logic. Despite the mild assumptions of the original theory, recent developments have shown that the institution-based approach cannot capture constructions that arise when service-oriented computing is presented as a form of logic programming, thus prompting the need for a new perspective on Herbrand's theorem founded instead upon a concept of generalized substitution system. In this paper, we formalize the connection between the institution- and the substitution-system-based approach to logic programming by investigating a number of features of institutions, like the existence of a quantification space or of representable substitutions, under which they give rise to suitable generalized substitution systems. Building on these results, we further show how the original institution-independent versions of Herbrand's theorem can be obtained as concrete instances of a more general result.

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1 Introduction

The Fundamental Theorem of Herbrand [15] is a central result in proof theory that deals with the reduction of provability in first-order logic to provability in propositional logic. Its importance in the context of automated theorem proving was realized in the early 1960s, when, in combination with the theory of Horn-clause logic, it played a key role in establishing the mathematical foundations of logic programming (see e.g. [16]). In the conventional setting of relational first-order logic, Herbrand's theorem states that, for a set Γ of Horn clauses (i.e. for a logic program Γ), the answers to an existential query can be found simply by examining a term model – called the (least) Herbrand model – instead of all the models that satisfy Γ. Over the last three decades, the original result has been generalized to a variety of other logical systems, including Horn-clause logic with equality [13, 14], hidden algebra [12], and category-based constraint logic [3], culminating in [5] with an investigation of Herbrand's theorem in an arbitrary institution [11] – a categorical formalization of the intuitive notion of logical system put forward by Goguen and Burstall in the late 1970s.

The results presented in [5] are grounded on an institution-independent treatment of variables as signature morphisms (which correspond in concrete cases to extensions of

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signatures with new constant-operation symbols) that was first outlined in [24]. This enabled the development of fundamental semantic concepts to logic programming like query and solution in arbitrary institutions. Logic programs, for example, are given by theory presentations (usually universal Horn presentations), while queries are captured through existential quantifications of basic sentences. We will recall these concepts together with their corresponding properties one step at a time in the subsequent sections of this paper.

Thanks to its generality, the institution-based approach to Herbrand's theorem enabled the development of logic programming over a wide range of logical systems (see e.g. [10], and also [6]). All the same, certain institution-based forms of logic programming do not fit into the framework proposed in [5]. In particular, the logic-programming semantics of services [30] is grounded on a family of logical systems for which the concept of variable cannot be faithfully captured by means of representable extensions of signatures, thus failing to meet one of the most basic assumptions of the institution-independent variant of the theorem. This led us to advance in [29] a new abstract approach to logic programming and, implicitly, to Herbrand's theorem over a concept of generalized substitution system that extends institutions by allowing for direct representations of variables and substitutions – similarly to the context institutions of [23], though the latter are concrete, in the sense that the category of models of every signature is concrete over the category of indexed sets.

In the present paper, we continue the work reported in [29] with an investigation of the relationship between the institution-based and the substitution-system-based approach to logic programming. More specifically, we show that the hypotheses of the latter are indeed more general by examining the role of representability (of signature morphisms in arbitrary institutions) in the construction of a generalized substitution system. The main challenge here lies in the treatment of substitutions, which, in the institutional setting, are captured purely through their corresponding translations of sentences and reductions of models. This prevents us from using the additional information available when substitutions are regarded as mappings from variables to terms – which is only possible, however, in concrete examples of institutions such as first-order logic – thus making it difficult to translate substitutions along signature morphisms. For this reason, the main contributions of our paper, namely the derivation of a generalized substitution system from a given institution and the reformulation of the original institution-independent variants of Herbrand's theorem in the resulting framework, are parameterized by a class of general substitutions.

The paper is organized as follows: in Section 2 we review the concept of generalized substitution system and two well-known formalisms that have been studied in the context of institution-based logic-programming languages; in Section 3 we examine a class of substitution systems whose variables are defined through extensions of signatures (of a given institution), and whose substitutions correspond to the institution-independent notion of substitution; building on these results, in Section 4 we further investigate the translation of variables along signature morphisms and identify a set of sufficient conditions under which an institution can give rise to a generalized substitution system; lastly, in Section 5 we present a different perspective on the institution-independent versions of Herbrand's fundamental theorem.

2 Technical preliminaries

We generally assume familiarity with the theory of institutions, including its categorical underpinnings and the presentation of institutions as functors into the category $\mathbb{R}$oom of rooms and corridors (see, for example, the monographs [6, 25]). In terms of category-theoretic notations, we denote by $|\mathbb{C}|$ the collection of objects of a category $\mathbb{C}$, by $f \fatsemi g$ the composition

of arrows f and g in diagrammatic order, and by 1_A the identity arrow of an object A.

Our work makes extensive use of comma categories. To this end, for any object A of a category $\mathbb{C}$, we denote the comma category of $\mathbb{C}$-objects under A by $A \,/\, \mathbb{C}$ and the forgetful functor $A \,/\, \mathbb{C} \to \mathbb{C}$ by $|_|_A$. We also denote by $\mathbb{C}^{\to}$ the category of $\mathbb{C}$-arrows, and by *dom* the canonical projection functor $\mathbb{C}^{\to} \to \mathbb{C}$ that maps the arrows $f\colon A \to B$ in $\mathbb{C}$ to their domain.

2.1 Generalized substitution systems

Substitution systems are the most basic structures that underlie both the denotational and the operational semantics of the logic-independent approach to logic programming proposed in [29]. Since their definition relies technically on the category $\mathbb{R}$oom, we start by recalling that a *room* is a triple $\langle S, \mathbb{M}, \models \rangle$ consisting of a set S of *sentences*, a category $\mathbb{M}$ of *models*, and a *satisfaction relation* $\models \subseteq |\mathbb{M}| \times S$. Furthermore, a *corridor* (i.e. a morphism of rooms) $\langle \alpha, \beta \rangle\colon \langle S, \mathbb{M}, \models \rangle \to \langle S', \mathbb{M}', \models' \rangle$ is defined by a *sentence-translation function* $\alpha\colon S \to S'$ and a *model-reduction functor* $\beta\colon \mathbb{M}' \to \mathbb{M}$ such that, for all $M' \in |\mathbb{M}'|$ and $\rho \in S$,

$$M' \models' \alpha(\rho) \qquad \text{if and only if} \qquad \beta(M') \models \rho.$$

The following definitions originate from [29].

▶ **Definition 1** (Substitution system)**.** A *substitution system* is a triple $\langle \mathbb{S}\text{ubst}, G, \mathcal{S} \rangle$, usually denoted simply by $\mathcal{S}$, that consists of

- a category $\mathbb{S}$ubst of (*abstract*) *signatures of variables* and *substitutions*,
- a room G of *ground sentences* and *models*, and
- a functor $\mathcal{S}\colon \mathbb{S}\text{ubst} \to G \,/\, \mathbb{R}\text{oom}$ defining, for every signature of variables X, the corridor $\mathcal{S}(X)\colon G \to G(X)$ from G to the room $G(X)$ of *X-sentences* and *X-models*.

A classical example can be obtained by defining $\mathbb{S}$ubst as the category of (sets of) variables and substitutions over a fixed first-order signature Σ. In that case, G is the room of those sentences of Σ that are ground (i.e. without variables), and the corridors $\mathcal{S}(X)$ correspond to the signature morphisms that extend Σ by adding the variables of X as new constants.

Generalized substitution systems can be regarded as extensions of substitution systems that are parameterized by the signature used. In this sense, the connection between generalized substitutions systems and substitution systems is similar to that between institutions and rooms: generalized substitutions systems are functors into the category $\mathbb{S}$ubst$\mathbb{S}$ys of substitutions systems. To make this definition more precise, we recall from [29] that a *morphism of substitution systems* between $\mathcal{S}\colon \mathbb{S}\text{ubst} \to G \,/\, \mathbb{R}\text{oom}$ and $\mathcal{S}'\colon \mathbb{S}\text{ubst}' \to G' \,/\, \mathbb{R}\text{oom}$ is a triple $\langle \Psi, \kappa, \tau \rangle$, where Ψ is a functor $\mathbb{S}\text{ubst} \to \mathbb{S}\text{ubst}'$, κ is a corridor $G \to G'$, and τ is a natural transformation $\mathcal{S} \Rightarrow \Psi \fatsemi \mathcal{S}' \fatsemi (\kappa \,/\, \mathbb{R}\text{oom})$.

▶ **Definition 2** (Generalized substitution system)**.** A *generalized substitution system* is a pair $\langle \mathbb{S}\text{ig}, \mathcal{GS} \rangle$ given by a category $\mathbb{S}$ig of *signatures* and a functor $\mathcal{GS}\colon \mathbb{S}\text{ig} \to \mathbb{S}\text{ubst}\mathbb{S}\text{ys}$.

2.2 Equational logic programming

Before we embark on the study of institution-based abstract logic-programming languages, let us briefly survey the logical systems that underlie two of the most prominent examples of (concrete) logic-programming languages examined in the context of institutions: first-order and higher-order equational logic programming (see e.g. [13, 19], and also [22]). These will form the main reference points that we will use to illustrate the various concepts and properties discussed in the subsequent sections of our paper.

First-order equational logic

First-order equational logic programming is defined over the quantifier-free fragment of many-sorted first-order equational logic, whose institution we denote by QF-FOL$_=$. In what follows, we only give a succinct presentation of the most important first-order concepts needed for the purpose of our work. A more in-depth discussion of QF-FOL$_=$ can be found, for example, in [14, 29], or in the recent monographs [6, 25].

Signatures. The *signatures* of QF-FOL$_=$ are pairs $\langle S, F\rangle$, where S is a (finite) set of *sorts* and F is a family $(F_{w\to s})_{w\in S^*, s\in S}$ of (finite) sets of *operation symbols* indexed by *arities* and *sorts*. *Signature morphisms* $\varphi\colon \langle S, F\rangle \to \langle S', F'\rangle$ are defined by functions $\varphi^{\mathrm{st}}\colon S \to S'$ between the sets of sorts and by families of functions $\varphi^{\mathrm{op}}_{w\to s}\colon F_{w\to s} \to F'_{\varphi^{\mathrm{st}}(w)\to\varphi^{\mathrm{st}}(s)}$, for $w \in S^*$ and $s \in S$, between the sets of operation symbols.

Sentences, models, and the satisfaction relation. For every signature $\langle S, F\rangle$ and every sort $s \in S$, the set $\mathrm{T}_{F,s}$ of *F-terms* of sort s is the least set such that $\sigma(t_1, \ldots, t_n)\colon s \in \mathrm{T}_{F,s}$ for all $\sigma \in F_{s_1\cdots s_n\to s}$ and $t_i \in \mathrm{T}_{F,s_i}$. The *sentences* over $\langle S, F\rangle$ are built from *equational atoms* $l = r$, where $l, r \in \mathrm{T}_{F,s}$ for some $s \in S$, by iteration of the usual Boolean connectives.

The *models*, or *algebras*, M of $\langle S, F\rangle$ interpret each sort $s \in S$ as a set M_s, called the *carrier* of s in M, and each operation symbol $\sigma \in F_{s_1\cdots s_n\to s}$ as a function $M_\sigma\colon M_{s_1} \times \cdots \times M_{s_n} \to M_s$. *Homomorphisms* $h\colon M_1 \to M_2$ are families of functions $(h_s\colon M_{1,s} \to M_{2,s})_{s\in S}$ such that $h_s(M_{1,\sigma}(m_1, \ldots, m_n)) = M_{2,\sigma}(h_{s_1}(m_1), \ldots, h_{s_n}(m_n))$ for all $\sigma \in F_{s_1\cdots s_n\to s}$ and $m_i \in M_{s_i}$.

Finally, the *satisfaction relation* is defined by induction on the structure of sentences, based on the evaluation of terms in models. For instance, an $\langle S, F\rangle$-model M satisfies an equational atom $l = r$ if and only if the terms l and r yield the same value in M.

Higher-order logic with Henkin semantics

Following the lines of [20], and also of more recent institution-theoretic works such as [25, 28], we define and study higher-order logic programming over a simplified version of higher-order logic with Henkin semantics that only takes into account λ-free terms. This does not limit the expressive power of the logic since for any term $\lambda(x\colon s).\, t$ one can define a new constant σ and a universal sentence of the form $\forall\{x\colon s\} \cdot \sigma\, x = t$.[1] Similarly to first-order equational logic programming, for the results presented in the following sections it suffices to consider the quantifier-free fragment of higher-order logic, whose institution we denote by QF-HNK.[2]

Signatures. A *higher-order signature* consists of a set S of *basic types*, or *sorts*, and a family $(F_s)_{s\in\vec{S}}$ of sets of *constant-operation symbols* indexed by *S-types*, where $\vec{S}$ is the least set for which $S \subseteq \vec{S}$ and $s_1 \to s_2 \in \vec{S}$ whenever $s_1, s_2 \in \vec{S}$. *Signature morphisms* $\varphi\colon \langle S, F\rangle \to \langle S', F'\rangle$ comprise functions $\varphi^{\mathrm{st}}\colon S \to \vec{S'}$ and $\varphi^{\mathrm{op}}_s\colon F_s \to F'_{\varphi^{\mathrm{type}}(s)}$, for $s \in \vec{S}$, where $\varphi^{\mathrm{type}}\colon \vec{S} \to \vec{S'}$ is the canonical extension of φ^{st} given by $\varphi^{\mathrm{type}}(s_1 \to s_2) = \varphi^{\mathrm{type}}(s_1) \to \varphi^{\mathrm{type}}(s_2)$.

Sentences, models, and the satisfaction relation. Given a signature $\langle S, F\rangle$, the family $\big(\mathrm{T}_{F,s}\big)_{s\in\vec{S}}$ of *F-terms* is the least family of sets such that $\sigma\colon s \in \mathrm{T}_{F,s}$ for all $s \in \vec{S}$ and $\sigma \in F_s$, and $(t\, t_1) \in \mathrm{T}_{F,s_2}$ for all terms $t \in \mathrm{T}_{F,s_1\to s_2}$ and $t_1 \in \mathrm{T}_{F,s_1}$. As in the case of QF-FOL$_=$, the *sentences* over $\langle S, F\rangle$ are built from *equational atoms* $l = r$, where l and r are terms in $\mathrm{T}_{F,s}$ for some type $s \in \vec{S}$, by repeated applications of the Boolean connectives.

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[1] A detailed presentation of this encoding, formalized as an institution comorphism, can be found in [9].

[2] Note that the universal sentences needed for encoding λ-terms can still be defined as Horn clauses of the logic programs under consideration.

The *models* M of a higher-order signature $\langle S, F\rangle$ interpret the types $s \in \vec{S}$ as sets M_s, the constant symbols $\sigma \in F_s$ as elements $M_\sigma \in M_s$, and define injective maps $[\![_]\!]^M_{s_1\to s_2} \colon M_{s_1\to s_2} \to [M_{s_1} \to M_{s_2}]$, where $[M_{s_1} \to M_{s_2}]$ denotes the set of functions from M_{s_1} to M_{s_2}, for any two types $s_1, s_2 \in \vec{S}$. *Model homomorphisms* $h \colon M_1 \to M_2$ are families of maps $(h_s \colon M_{1,s} \to M_{2,s})_{s\in\vec{S}}$ such that $h_s(M_{1,\sigma}) = M_{2,\sigma}$ for every type $s \in \vec{S}$ and operation $\sigma \in F_s$, and $[\![f]\!]^{M_1}_{s_1\to s_2} \fatsemi h_{s_2} = h_{s_1} \fatsemi [\![h_{s_1\to s_2}(f)]\!]^{M_2}_{s_1\to s_2}$ for all $s_1, s_2 \in \vec{S}$ and $f \in M_{1,s_1\to s_2}$.

The *satisfaction relation* relies once again on the interpretation of terms in models, which extends the interpretation of constant-operation symbols as follows: for every $\langle S, F\rangle$-model M and every pair of terms $t \in \mathrm{T}_{F,s_1\to s_2}$ and $t_1 \in \mathrm{T}_{F,s_1}$, $M_{(t\,t_1)} = [\![M_t]\!]^M_{s_1\to s_2}(M_{t_1})$.

3 Institution-independent substitutions

The institution-independent concept of substitution (see [5], and also [6]) generalizes first-order substitutions (as well as second-order and higher-order substitutions, among others) to arbitrary institutions by taking notice only of their syntactic and semantic effects: the translations of sentences and the reductions of models that they generate. A key step in arriving at this notion is the presentation of the extensions of signatures by sets of variables as particular cases of signature morphisms (along the lines of [24]). Thus, for any two signature morphisms $\chi_1 \colon \Sigma \to \Sigma_1$ and $\chi_2 \colon \Sigma \to \Sigma_2$ (two extensions of Σ) in an institution $\mathcal{I} = \langle \mathbb{S}\mathrm{ig}, \mathrm{Sen}, \mathrm{Mod}, \models\rangle$, a *substitution* $\psi \colon \chi_1 \to \chi_2$ is a pair $\langle \mathrm{Sen}_\Sigma(\psi), \mathrm{Mod}_\Sigma(\psi)\rangle$ given by

- a sentence-translation function $\mathrm{Sen}_\Sigma(\psi) \colon \mathrm{Sen}(\Sigma_1) \to \mathrm{Sen}(\Sigma_2)$ and
- a model-reduction functor $\mathrm{Mod}_\Sigma(\psi) \colon \mathrm{Mod}(\Sigma_2) \to \mathrm{Mod}(\Sigma_1)$

that preserve Σ, in the sense that $\mathrm{Sen}(\chi_1) \fatsemi \mathrm{Sen}_\Sigma(\psi) = \mathrm{Sen}(\chi_2)$ and $\mathrm{Mod}_\Sigma(\psi) \fatsemi \mathrm{Mod}(\chi_1) = \mathrm{Mod}(\chi_2)$, and satisfy the following condition:

$$M_2 \models_{\Sigma_2} \mathrm{Sen}_\Sigma(\psi)(\rho_1) \qquad \text{if and only if} \qquad \mathrm{Mod}_\Sigma(\psi)(M_2) \models_{\Sigma_1} \rho_1$$

for every Σ_2-model M_2 and every Σ_1-sentence ρ_1.

In this work, we take into consideration an equivalent formulation of the original definition that makes use of the category $\mathbb{R}\mathrm{oom}$ of rooms and corridors. In addition, we extend the fact that substitutions inherit the composition of their components – thus giving rise to a category – to derive a general substitution system of Σ-substitutions for each signature Σ.

▶ **Proposition 3.** *Let $\mathcal{Q} \subseteq \mathbb{S}\mathrm{ig}$ be a class of signature morphisms of an institution $\mathcal{I}$ regarded as a functor $\mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$. For every $\mathcal{I}$-signature Σ we obtain a substitution system $\mathcal{SI}^{\mathcal{Q}}_\Sigma \colon \mathbb{S}\mathrm{ubst}^{\mathcal{Q}}_\Sigma \to \mathcal{I}(\Sigma) \,/\, \mathbb{R}\mathrm{oom}$ defined as follows:*

- *The objects of the category $\mathbb{S}\mathrm{ubst}^{\mathcal{Q}}_\Sigma$ – i.e. the* signatures of Σ-variables *– are signature morphisms $\chi \colon \Sigma \to \Sigma(\chi)$[3] belonging to the class $\mathcal{Q}$. Their corresponding corridors via the functor $\mathcal{SI}^{\mathcal{Q}}_\Sigma$ are given simply by $\mathcal{I}(\chi) \colon \mathcal{I}(\Sigma) \to \mathcal{I}(\Sigma(\chi))$.*
- *For every two signatures of Σ-variables $\chi_1 \colon \Sigma \to \Sigma(\chi_1)$ and $\chi_2 \colon \Sigma \to \Sigma(\chi_2)$, a* Σ-substitution *$\psi \colon \chi_1 \to \chi_2$ (i.e. a morphism in $\mathbb{S}\mathrm{ubst}^{\mathcal{Q}}_\Sigma$) consists of a corridor $\langle \mathrm{Sen}_\Sigma(\psi), \mathrm{Mod}_\Sigma(\psi)\rangle$ between $\mathcal{I}(\Sigma(\chi_1))$ and $\mathcal{I}(\Sigma(\chi_2))$ such that $\mathcal{I}(\chi_1) \fatsemi \langle \mathrm{Sen}_\Sigma(\psi), \mathrm{Mod}_\Sigma(\psi)\rangle = \mathcal{I}(\chi_2)$.*

$$\mathcal{I}(\Sigma(\chi_1)) \xleftarrow{\mathcal{I}(\chi_1)} \mathcal{I}(\Sigma) \xrightarrow{\mathcal{I}(\chi_2)} \mathcal{I}(\Sigma(\chi_2)) \qquad \langle \mathrm{Sen}_\Sigma(\psi), \mathrm{Mod}_\Sigma(\psi)\rangle : \mathcal{I}(\Sigma(\chi_1)) \to \mathcal{I}(\Sigma(\chi_2))$$

[3] For convenience, we denote the codomain of the signature morphism χ by $\Sigma(\chi)$; this reflects the intuition that $\Sigma(\chi)$ is an extension of the signature Σ with variables defined by χ.

As such, Σ-substitutions are merely arrows in the comma category $\mathcal{I}(\Sigma) \,/\, \mathbb{R}\mathrm{oom}$, meaning that they are identified with their images under the functor $\mathcal{SI}^{\mathcal{Q}}_{\Sigma}$. The composition of substitutions is defined accordingly. ◀

▶ **Example 4.** In QF-FOL$_=$, a *variable* over a signature $\langle S, F\rangle$ is a triple $(x, s, F_{\varepsilon\to s})$,[4] often denoted simply by $x\colon s$, where x is the name of the variable and s is its sort. Thus, QF-FOL$_=$-signatures of $\langle S, F\rangle$-variables X are S-indexed families of sets X_s of variables of sort s such that different variables have different names. *First-order substitutions* $\psi\colon X \to Y$ can be further defined as S-indexed families of maps $\psi_s\colon X_s \to \mathrm{T}_{F\cup Y,s}$ that assign a term over the extended signature $\langle S, F \cup Y\rangle$ to every variable of X.

One can easily check that first-order substitutions $\psi\colon X \to Y$ indeed give rise to general substitutions between $\langle S, F\rangle \subseteq \langle S, F\cup X\rangle$ and $\langle S, F\rangle \subseteq \langle S, F\cup Y\rangle$ (see e.g. [6]). For instance, the reduct $\mathrm{Mod}_{\langle S,F\rangle}(\psi)(N)$ of an $\langle S, F\cup Y\rangle$-model N is the $\langle S, F\cup X\rangle$-expansion of $N\restriction_{\langle S,F\rangle}$ given by $\mathrm{Mod}_{\langle S,F\rangle}(\psi)(N)_{x\colon s} = N_{\psi(x)}$ for every variable $x\colon s$ of X. Note, however, that not every general substitution between $\langle S, F\rangle \subseteq \langle S, F\cup X\rangle$ and $\langle S, F\rangle \subseteq \langle S, F\cup Y\rangle$ corresponds to a first-order substitution; we will discuss this aspect to a greater extent in Section 4.2.

Higher-order signatures of variables and *substitutions* can be defined likewise, by recalling that a *higher-order variable* over a signature $\langle S, F\rangle$ is a triple of the form (x, s, F_s), where x and s correspond to the name and the type of the variable (see e.g. [28], and also [2]). As expected, this means that we allow higher-order variables to range over arbitrary functions.

4 Quantification spaces

Due to its mild assumptions, the construction outlined in Proposition 3 cannot be easily generalized to accommodate signature morphisms. More precisely, one cannot guarantee that signature morphisms $\varphi\colon \Sigma \to \Sigma'$ lead to adequate morphisms between the substitution systems associated with Σ and Σ': it would suffice, for example, to consider a class $\mathcal{Q}$ of signature morphisms that consists only of extensions of Σ, thus preventing the translation of the signatures of Σ-variables along φ. To overcome this limitation, we take into account only those extensions of signatures that belong to a quantification space – a notion introduced in [7] in the context of quasi-Boolean encodings[5] and utilized in a series of papers on hybridization and many-valued institutions (see e.g. [17, 8]). For the purpose of our work, it will be convenient to consider a more categorical formulation of the original definition of quantification spaces, based on Fact 5 below.

▶ Fact 5. Consider a category $\mathbb{C}$ and a subcategory $\mathbb{Q}$ of the category $\mathbb{C}^{\to}$ of $\mathbb{C}$-arrows. The domain functor $dom\colon \mathbb{Q} \to \mathbb{C}$ gives rise to a natural transformation $\iota_{\mathbb{Q}}\colon (_ \,/\, \mathbb{Q}) \Rightarrow dom^{\mathrm{op}} \fatsemi (_ \,/\, \mathbb{C})$ where $(_ \,/\, \mathbb{Q})\colon \mathbb{Q}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ and $(_ \,/\, \mathbb{C})\colon \mathbb{C}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ are the canonical comma-category functors and, for every triple $\langle A_1, f, A_2\rangle$ in $|\mathbb{Q}|$ (i.e. arrow $f\colon A_1 \to A_2$ in $\mathbb{C}$), $\iota_{\mathbb{Q},f}\colon f \,/\, \mathbb{Q} \to A_1 \,/\, \mathbb{C}$ is the functor that maps the morphisms $\langle g_1, g_2\rangle\colon \langle A_1, f, A_2\rangle \to \langle A'_1, f', A'_2\rangle$ in $\mathbb{Q}$ (corresponding to commutative squares in $\mathbb{C}$) to $g_1\colon A_1 \to A'_1$.

$$\begin{array}{ccc} A_1 & \xrightarrow{g_1} & A'_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A_2 & \xrightarrow[g_2]{} & A'_2 \end{array} \quad \mapsto \quad A_1 \xrightarrow{g_1} A'_1$$

[4] We denote the empty arity by ε; hence, $F_{\varepsilon\to s}$ is the set of *constants* of sort s of the signature $\langle S, F\rangle$.
[5] It should be noted, however, that the ideas that underlie quantification spaces can be traced back to [27] – one of the earliest works in which open formulae are treated in arbitrary institutions.

▶ **Definition 6** (Quantification space)**.** For any institution $\mathcal{I}\colon \mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$, a *quantification space* consists of a subcategory $\mathbb{Q}$ of $\mathbb{S}\mathrm{ig}^{\to}$ such that

1. every arrow in $\mathbb{Q}$ corresponds to a pushout in $\mathbb{S}\mathrm{ig}$, and
2. the transformation $\iota_{\mathbb{Q}}\colon (_ \,/\, \mathbb{Q}) \Rightarrow dom^{\mathrm{op}} \fatsemi (_ \,/\, \mathbb{S}\mathrm{ig})$ is a natural isomorphism.

This means that for every extension of signatures $\chi\colon \Sigma \to \Sigma(\chi)$ in $|\mathbb{Q}|$ and every signature morphism $\varphi\colon \Sigma \to \Sigma'$ there exist a unique extension $\chi'\colon \Sigma' \to \Sigma'(\chi')$ in $|\mathbb{Q}|$ and a unique signature morphism $\phi\colon \Sigma(\chi) \to \Sigma'(\chi')$ such that the pair $\langle \varphi, \phi \rangle$ defines a morphism in $\mathbb{Q}$ between the arrows χ and χ'.[6] We will henceforth denote the signature extension χ' and the signature morphism ϕ by $\chi^{\varphi}\colon \Sigma' \to \Sigma'(\chi^{\varphi})$ and $\varphi^{\chi}\colon \Sigma(\chi) \to \Sigma'(\chi^{\varphi})$, respectively.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi^{\varphi}} \\ \Sigma(\chi) & \xrightarrow[\varphi^{\chi}]{} & \Sigma'(\chi^{\varphi}) \end{array}$$

▶ **Example 7.** For both QF-FOL$_=$ and QF-HNK, the extensions of signatures $\chi\colon \langle S, F\rangle \to \langle S, F \cup X\rangle$ defined by (families of) finite sets of first-order/higher-order $\langle S, F\rangle$-variables X form a quantification space. More precisely, for every signature morphism $\varphi\colon \langle S, F\rangle \to \langle S', F'\rangle$,

- $\chi^{\varphi}\colon \langle S', F'\rangle \to \langle S', F' \cup X^{\varphi}\rangle$ is the extension of $\langle S', F'\rangle$ given by the sets of variables $X^{\varphi}_{s'} = \{x\colon s' \mid x\colon s \in X_s \text{ for some sort } s \in S \text{ (or type } s \in \vec{S}) \text{ such that } \varphi(s) = s'\}$, and
- $\varphi^{\chi}\colon \langle S, F \cup X\rangle \to \langle S', F' \cup X^{\varphi}\rangle$ is the canonical extension of φ that maps each $\langle S, F\rangle$-variable $x\colon s$ in X to the $\langle S', F'\rangle$-variable $x\colon \varphi(s)$ in X^{φ}.

▶ **Definition 8** (Adequacy)**.** For any institution, a quantification space $\mathbb{Q}$ is said to be *adequate* if every arrow $\langle \varphi, \varphi^{\chi}\rangle\colon \chi \to \chi^{\varphi}$ in $\mathbb{Q}$ corresponds to a model-amalgamation square: for every Σ'-model M' and $\Sigma(\chi)$-model N such that $M'{\restriction_{\varphi}} = N{\restriction_{\chi}}$ there exists a unique model N' of $\Sigma'(\chi^{\varphi})$ – the *amalgamation* of M' and N – such that $N'{\restriction_{\chi^{\varphi}}} = M'$ and $N'{\restriction_{\varphi^{\chi}}} = N$.

In semi-exact institutions[7] – like QF-FOL$_=$ (see [18]) – all quantification spaces are adequate. This is not the case of QF-HNK, for which it is known that, due to the presence of higher-order types, not every pushout square of signature morphisms is a model-amalgamation square (see e.g. [2]). Nonetheless, the quantification space for QF-HNK outlined in Example 7 is adequate: the amalgamation N' of any two given models M' of $\langle S', F'\rangle$ and N of $\langle S, F \cup X\rangle$ is the unique χ^{φ}-expansion of M' that satisfies $N'_{x\colon \varphi(s)} = N_{x\colon s}$ for each variable $x\colon s$ in X.

▶ Remark 9. Since the morphisms of any quantification space $\mathbb{Q}$ are required to form a category (by definition, $\mathbb{Q}$ is a subcategory of $\mathbb{S}\mathrm{ig}^{\to}$), for every signature extension $\chi\colon \Sigma \to \Sigma(\chi)$ in $|\mathbb{Q}|$ and every pair of composable signature morphisms $\varphi\colon \Sigma \to \Sigma'$ and $\varphi'\colon \Sigma' \to \Sigma''$, we have $(\chi^{\varphi})^{\varphi'} = \chi^{\varphi \fatsemi \varphi'}$ and $\varphi^{\chi} \fatsemi (\varphi')^{\chi^{\varphi}} = (\varphi \fatsemi \varphi')^{\chi}$. Moreover, $\chi^{1_{\Sigma}} = \chi$ and $1^{\chi}_{\Sigma} = 1_{\Sigma(\chi)}$.

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\varphi} & \Sigma' & \xrightarrow{\varphi'} & \Sigma'' \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi^{\varphi}} & & \downarrow{\scriptstyle(\chi^{\varphi})^{\varphi'}=\chi^{\varphi\fatsemi\varphi'}} \\ \Sigma(\chi) & \xrightarrow{\varphi^{\chi}} & \Sigma'(\chi^{\varphi}) & \xrightarrow{(\varphi')^{\chi^{\varphi}}} & \Sigma''((\chi^{\varphi})^{\varphi'}) \\ & {\scriptstyle(\varphi\fatsemi\varphi')^{\chi}}\searrow & & \nearrow & \end{array}$$

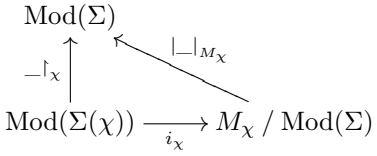
Quantification spaces thus provide adequate support for translating abstract signature extensions along morphisms of signatures in a functorial manner.

[6] Moreover, the signature morphisms χ' and ϕ correspond to a pushout of φ and χ.
[7] We recall that an institution is semi-exact if its model functor preserves pullbacks.

4.1 Representable signature extensions

Since the institution-independent substitutions of Proposition 3 correspond to a semantic concept, we cannot expect to translate them along signature morphisms in the same manner as the extensions of signatures. The solution that we propose herein relies on an important characterization of the first-order signature extensions with new constant-operation symbols: for every QF-FOL$_=$-signature extension $\langle S, F\rangle \subseteq \langle S, F \cup X\rangle$ there is a one-to-one correspondence between the models of $\langle S, F \cup X\rangle$ and the model homomorphisms defined on the free $\langle S, F\rangle$-algebra $\mathrm{T}_F(X)$ over the set of variables X; in particular, every $\langle S, F \cup X\rangle$-model N determines the homomorphism $h\colon \mathrm{T}_F(X) \to N{\restriction_{\langle S,F\rangle}}$ given by $h(x) = N_x$ for every variable x in X. In this context, $\mathrm{T}_F(X)$ is said to be a *representation* of the inclusion of signatures $\langle S, F\rangle \subseteq \langle S, F \cup X\rangle$. The following definition originates from [4].

▶ **Definition 10** (Representable signature morphism)**.** In any institution, a signature morphism $\chi\colon \Sigma \to \Sigma(\chi)$ is *representable* if there exist a Σ-model M_χ, called the *representation* of χ, and an isomorphism of categories i_χ between $\mathrm{Mod}(\Sigma(\chi))$ and $M_\chi \,/\, \mathrm{Mod}(\Sigma)$ such that the following diagram commutes.



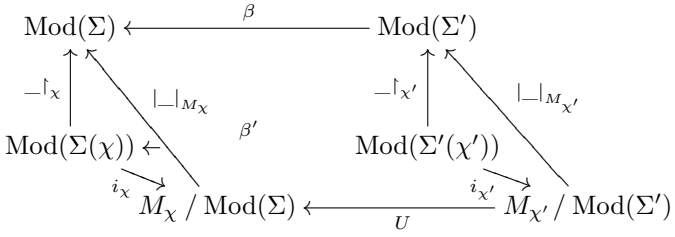
Representable first-order signature morphisms were studied in depth in [26], from where we recall Proposition 11 below (see also [6]).

▶ **Proposition 11.** *A first-order signature morphism is representable if and only if it is bijective on all symbols, except constant-operation symbols.* ◀

Consequently, all QF-FOL$_=$-signature extensions with constants are representable.

A similar result can be obtained for QF-HNK. In that case, however, the signature extensions with constants $\langle S, F\rangle \subseteq \langle S, F \cup X\rangle$ can only be guaranteed to be quasi-representable, in the sense that, for every $\langle S, F \cup X\rangle$-model N, the canonical functor $N \,/\, \mathrm{Mod}(S, F \cup X) \to N{\restriction_{\langle S,F\rangle}} \,/\, \mathrm{Mod}(S, F)$ determined by the model-reduct functor $_{\restriction_{\langle S,F\rangle}}$ is an isomorphism (see, for example, [2] for more details). Representability further requires that the resulting higher-order signatures $\langle S, F \cup X\rangle$ have initial models, a property which holds whenever $\langle S, F \cup X\rangle$ has at least one constant-operation symbol for each type.[8]

▶ Remark 12. Let $\chi\colon \Sigma \to \Sigma(\chi)$ and $\chi'\colon \Sigma' \to \Sigma'(\chi')$ be a pair of representable signature extensions defined by a quantification space $\mathbb{Q}$, and let β and β' be two functors as depicted below such that $\mathrm{Mod}(\chi') \fatsemi \beta = \beta' \fatsemi \mathrm{Mod}(\chi)$.



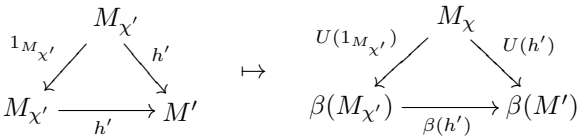
[8] This property is commonly achieved by assuming that, for each type $s \in \vec{S}$, the set F_s contains an implicit constant-operation symbol.

The composition $i_{\chi'}^{-1} \fatsemi \beta' \fatsemi i_\chi$ gives rise to a functor U between the comma categories $M_{\chi'} \,/\, \mathrm{Mod}(\Sigma')$ and $M_\chi \,/\, \mathrm{Mod}(\Sigma)$, where

- for every Σ'-model homomorphism $h' \colon M_{\chi'} \to M'$, $U(h')$ is the Σ-model homomorphism $(i_{\chi'}^{-1} \fatsemi \beta' \fatsemi i_\chi)(h') \colon M_\chi \to \beta(M')$, and
- for every arrow $f' \colon h'_1 \to h'_2$ between model homomorphisms $h'_1 \colon M_{\chi'} \to M'_1$ and $h'_2 \colon M_{\chi'} \to M'_2$, $U(f')$ is just the β-reduct of f', $\beta(f') \colon \beta(M'_1) \to \beta(M'_2)$.

When β is the model-reduct functor $\mathrm{Mod}(\varphi)$ of a signature morphism $\varphi \colon \Sigma \to \Sigma'$, and when χ' and β' are χ^φ and $\mathrm{Mod}(\varphi^\chi)$, respectively, we will denote the functor $U \colon M_{\chi'} \,/\, \mathrm{Mod}(\Sigma') \to M_\chi \,/\, \mathrm{Mod}(\Sigma)$ by $U_{\varphi,\chi}$. Similarly, when β is the identity of $\mathrm{Mod}(\Sigma)$ and β' is the underlying model functor of a substitution $\psi \colon \chi \to \chi'$, we will denote the functor U by U_ψ.

Model homomorphisms $h' \colon M_{\chi'} \to M'$ can be regarded both as objects and as arrows (between $1_{M_{\chi'}}$ and h') in the comma category $M_{\chi'} \,/\, \mathrm{Mod}(\Sigma')$. In combination with the definition of $U \colon M_{\chi'} \,/\, \mathrm{Mod}(\Sigma') \to M_\chi \,/\, \mathrm{Mod}(\Sigma)$, the arrow view provides us a useful factorization of $U(h')$ as $U(1_{M_{\chi'}}) \fatsemi \beta(h')$.



▶ Fact 13. Under the notation and hypotheses of Remark 12, for every Σ'-model homomorphism $h' \colon M_{\chi'} \to M'$, $U(h') = U(1_{M_{\chi'}}) \fatsemi \beta(h')$.

4.2 Representable substitutions

Quantification spaces that have representable extensions of signatures, meaning that every extension $\chi \colon \Sigma \to \Sigma(\chi)$ defined by the quantification space is representable, allow us to extend the concept of representability from signature extensions (i.e. signatures of variables) to substitutions, leading to a purely model-theoretic view of the categories of substitutions.

▶ **Proposition 14.** *For any signature Σ in an institution with a quantification space $\mathbb{Q}$, the representation of signature extensions generalizes to a functor $R_\Sigma^{\mathbb{Q}} \colon \mathbb{S}\mathrm{ubst}_\Sigma^{\mathbb{Q}} \to \mathrm{Mod}(\Sigma)$, where*

- *for every extension of signatures $\chi \colon \Sigma \to \Sigma(\chi)$ in $|\mathbb{Q}|$, $R_\Sigma^{\mathbb{Q}}(\chi) = M_\chi$, and*
- *for every substitution $\psi \colon \chi_1 \to \chi_2$, $R_\Sigma^{\mathbb{Q}}(\psi) = U_\psi(1_{M_{\chi_2}}) \colon M_{\chi_1} \to M_{\chi_2}$.*

Moreover, for every Σ-substitution ψ, $\mathrm{Mod}_\Sigma(\psi)$ is uniquely determined by $R_\Sigma^{\mathbb{Q}}(\psi)$. ◀

When the quantification space $\mathbb{Q}$ and the signature Σ are clear from the context, we may also denote the representation $R_\Sigma^{\mathbb{Q}}(\psi)$ of a substitution $\psi \colon \chi_1 \to \chi_2$ by $h_\psi \colon M_{\chi_1} \to M_{\chi_2}$.

Note that, in general, the functor $R_\Sigma^{\mathbb{Q}} \colon \mathbb{S}\mathrm{ubst}_\Sigma^{\mathbb{Q}} \to \mathrm{Mod}(\Sigma)$ of Proposition 14 need be neither full nor faithful. For example, in the case of QF-FOL$_=$, for every general substitution ψ we can define another substitution ψ' (with the same domain and codomain as ψ) such that, for every atomic sentence $l = r$ that is not ground, $\mathrm{Sen}_\Sigma(\psi')(l = r)$ corresponds to $\mathrm{Sen}_\Sigma(\psi)(r = l)$. In consequence, we can obtain distinct institution-independent substitutions having the same underlying model functor – and thus, the same representation. This is contrary to our intuition concerning first-order substitutions, where, given a signature $\langle S, F \rangle$, every substitution $\psi \colon X_1 \to X_2$, i.e. every S-indexed family of maps $\psi_s \colon X_{1,s} \to \mathrm{T}_{F \cup X_2, s}$, is determined uniquely by its representation $R_{\langle S,F \rangle}^{\mathbb{Q}}(\psi) \colon \mathrm{T}_F(X_1) \to \mathrm{T}_F(X_2)$. As we will see later, the full and faithful representation of substitutions as model homomorphisms is essential for translating substitutions along signature morphisms.

▶ **Definition 15** (Representable substitution)**.** Let Σ be a signature in an institution equipped with a quantification space $\mathbb{Q}$. For every subcategory $\mathbb{S}\mathrm{ubst}_\Sigma \subseteq \mathbb{S}\mathrm{ubst}^{\mathbb{Q}}_\Sigma$, a substitution $\psi\colon \chi_1 \to \chi_2$ in $\mathbb{S}\mathrm{ubst}_\Sigma$ is said to be $\mathbb{Q}$-*representable* if it is uniquely determined by its image under $R^{\mathbb{Q}}_\Sigma$. In addition, $\mathbb{S}\mathrm{ubst}_\Sigma$ forms a category of $\mathbb{Q}$-*representable* Σ-*substitutions* if the restriction of the functor $R^{\mathbb{Q}}_\Sigma\colon \mathbb{S}\mathrm{ubst}^{\mathbb{Q}}_\Sigma \to \mathrm{Mod}(\Sigma)$ to $\mathbb{S}\mathrm{ubst}_\Sigma$ is both full and faithful.

▶ **Example 16.** Let $\mathbb{Q}$ be the quantification space for QF-$\underline{\mathrm{FOL}}_=$ presented in Example 7. For every signature $\langle S, F\rangle$, the subcategory $\mathbb{S}\mathrm{ubst}_{\langle S,F\rangle} \subseteq \mathbb{S}\mathrm{ubst}^{\mathbb{Q}}_{\langle S,F\rangle}$ whose arrows correspond to the corridors induced by first-order substitution forms a category of $\mathbb{Q}$-representable substitutions. A similar property can be formulated for higher-order substitutions.

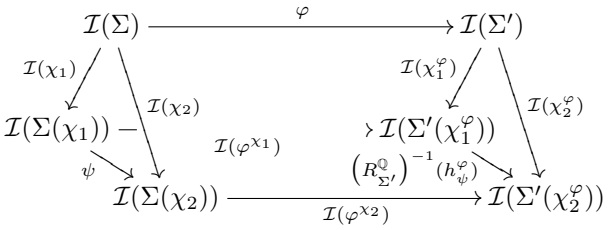
For the remaining part of this paper we will assume that $\mathcal{I}$ is an arbitrary but fixed institution $\langle \mathbb{S}\mathrm{ig}, \mathrm{Sen}, \mathrm{Mod}, \models\rangle$ equipped with

- *an adequate quantification space $\mathbb{Q} \subseteq \mathbb{S}\mathrm{ig}^{\to}$ of representable signature extensions, and*
- *a broad subcategory $\mathbb{S}\mathrm{ubst}_\Sigma \subseteq \mathbb{S}\mathrm{ubst}^{\mathbb{Q}}_\Sigma$ (i.e. with the same objects as $\mathbb{S}\mathrm{ubst}^{\mathbb{Q}}_\Sigma$), for every signature $\Sigma \in |\mathbb{S}\mathrm{ig}|$, of $\mathbb{Q}$-representable Σ-substitutions.*

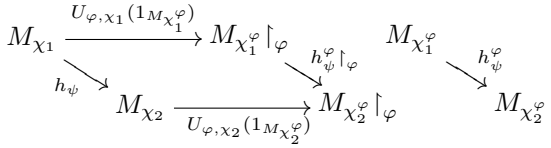
▶ **Lemma 17.** *Under the above assumptions, for every morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}\mathrm{ig}$ and every signature extension $\chi\colon \Sigma \to \Sigma(\chi)$ in $|\mathbb{Q}|$, the homomorphism $U_{\varphi,\chi}(1_{M_{\chi^\varphi}})\colon M_\chi \to M_{\chi^\varphi}{\restriction_\varphi}$ is a universal arrow from M_χ to $\mathrm{Mod}(\varphi)$.* ◀

The lemma above enables us to make use of a well-known construction of adjoint functors from universal arrows (see e.g. [1]) to derive translations between categories of substitutions.

▶ **Proposition 18.** *Every morphism of signatures $\varphi\colon \Sigma \to \Sigma'$ gives rise to a functor $\Psi_\varphi\colon \mathbb{S}\mathrm{ubst}_\Sigma \to \mathbb{S}\mathrm{ubst}_{\Sigma'}$ that maps*



- *every signature extension $\chi\colon \Sigma \to \Sigma(\chi)$ to $\chi^\varphi\colon \Sigma' \to \Sigma'(\chi^\varphi)$, and*
- *every Σ-substitution $\psi\colon \chi_1 \to \chi_2$ to $\psi^\varphi = \left(R^{\mathbb{Q}}_{\Sigma'}\right)^{-1}(h^\varphi_\psi)$, where h^φ_ψ is the unique Σ'-homomorphism $M_{\chi_1^\varphi} \to M_{\chi_2^\varphi}$ for which the diagram below commutes.*



Moreover, Ψ itself is functorial, in the sense that $\Psi_{\varphi;\varphi'} = \Psi_\varphi ; \Psi_{\varphi'}$ for every pair of composable signature morphisms $\varphi\colon \Sigma \to \Sigma'$ and $\varphi'\colon \Sigma' \to \Sigma''$, and $\Psi_{1_\Sigma} = 1_{\mathbb{S}\mathrm{ubst}_\Sigma}$.

Proof. The first part of the statement follows by Lemma 17 as a direct consequence of the universal property of the homomorphism $U_{\varphi,\chi_1}(1_{M_{\chi_1^\varphi}})$ – note that, since the functor $R^{\mathbb{Q}}_\Sigma$ is assumed to be both full and faithful, for every signature Σ, it suffices to reason about the representations of substitutions. With respect to the second part of the statement, notice first that, by the definition of quantification spaces, the translation of signature extensions

along signature morphisms is functorial (see Remark 9). In addition, by Remark 12, for every signature extension $\chi\colon \Sigma \to \Sigma(\chi)$, $U_{\varphi;\varphi',\chi} = U_{\varphi',\chi^\varphi} ; U_{\varphi,\chi}$. This allows us to deduce, according to Fact 13, that $U_{\varphi;\varphi',\chi}(1_{M_{\chi^\varphi}}) = U_{\varphi,\chi}(1_{M_{\chi^\varphi}});U_{\varphi',\chi^\varphi}(1_{M_{\chi^{\varphi;\varphi'}}})$. Hence, by the general properties of composing universal arrows, we can further conclude that the translation of substitutions along signature morphisms is also functorial. ◀

4.3 Deriving generalized substitution systems

For any signature morphism $\varphi\colon \Sigma \to \Sigma'$, the functor $\Psi_\varphi\colon \mathbb{S}\mathrm{ubst}_\Sigma \to \mathbb{S}\mathrm{ubst}_{\Sigma'}$ discussed in Proposition 18 can be extended in a straightforward manner to a morphism $\langle \Psi_\varphi, \kappa_\varphi, \tau_\varphi \rangle$ between the substitution systems $\mathcal{SI}_\Sigma$ and $\mathcal{SI}_{\Sigma'}$ obtained by restricting the functors $\mathcal{SI}^{\mathbb{Q}}_\Sigma$ and $\mathcal{SI}^{\mathbb{Q}}_{\Sigma'}$ of Proposition 3 to the subcategories $\mathbb{S}\mathrm{ubst}_\Sigma$ and $\mathbb{S}\mathrm{ubst}_{\Sigma'}$ of Σ- and Σ'-substitutions.

$$\begin{array}{ccc} \mathbb{S}\mathrm{ubst}_\Sigma & \xrightarrow{\mathcal{SI}_\Sigma} & \mathcal{I}(\Sigma)\,/\,\mathbb{R}\mathrm{oom} \\ {\scriptstyle \Psi_\varphi}\downarrow & \Downarrow {\scriptstyle \tau_\varphi} & \uparrow{\scriptstyle \mathcal{I}(\varphi)/\mathbb{R}\mathrm{oom}} \\ \mathbb{S}\mathrm{ubst}_{\Sigma'} & \xrightarrow[\mathcal{SI}_{\Sigma'}]{} & \mathcal{I}(\Sigma')\,/\,\mathbb{R}\mathrm{oom} \end{array}$$

To be more specific, κ_φ is the corridor $\langle \mathrm{Sen}(\varphi), \mathrm{Mod}(\varphi)\rangle$ obtained by taking the image $\mathcal{I}(\varphi)$ of φ under the institution $\mathcal{I}$, regarded as a functor into $\mathbb{R}\mathrm{oom}$. Furthermore, for every signature extension $\chi\colon \Sigma \to \Sigma(\chi)$, the corridor $\tau_{\varphi,\chi}\colon \mathcal{I}(\Sigma(\chi)) \to \mathcal{I}(\Sigma'(\chi^\varphi))$ is simply $\mathcal{I}(\varphi^\chi)$. It should be noted, however, that the naturality of τ_φ holds in general only up to semantic equivalence (see Proposition 19 below): this means that we can only guarantee that $\mathrm{Mod}(\varphi^\chi)$ is natural in χ, and thus that, for every substitution $\psi\colon \chi_1 \to \chi_2$ and sentence ρ over $\Sigma(\chi_1)$, $\varphi^{\chi_2}(\psi(\rho))$ and $\psi^\varphi(\varphi^{\chi_1}(\rho))$ are satisfied by the same class of models. In concrete cases like QF-FOL$_=$, the equality $\varphi^{\chi_2}(\psi(\rho)) = \psi^\varphi(\varphi^{\chi_1}(\rho))$ is usually due to the careful choice of the categories of substitutions; other choices, which may involve, for example, swapping the left- and the right-hand side of non-ground equational atoms, do not necessarily give rise to natural transformations τ_φ. For this reason, in what follows, we will implicitly assume that the categories $\mathbb{S}\mathrm{ubst}_\Sigma$ of Σ-substitutions are *compatible* with respect to signature morphisms, meaning that $\psi ; \mathcal{I}(\varphi^{\chi_2}) = \mathcal{I}(\varphi^{\chi_1}) ; \psi^\varphi$ for every substitution $\psi\colon \chi_1 \to \chi_2$.

▶ **Proposition 19.** *For every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every Σ-substitution $\psi\colon \chi_1 \to \chi_2$,* $\mathrm{Mod}(\varphi^{\chi_2}) ; \mathrm{Mod}_\Sigma(\psi) = \mathrm{Mod}_{\Sigma'}(\psi^\varphi) ; \mathrm{Mod}(\varphi^{\chi_1})$. ◀

We can now conclude the construction of a generalized substitution system $\mathcal{SI}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$ from an arbitrary institution $\mathcal{I}\colon \mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$ that satisfies the hypotheses laid out in Section 4.2 by noticing that, according to Proposition 18, to the fact that $\mathcal{I}$ is a functor, and to Remark 9, all components of the morphism of substitution systems $\langle \Psi_\varphi, \kappa_\varphi, \tau_\varphi \rangle$ presented above are functorial in φ. Moreover, since the quantification space of $\mathcal{I}$ is assumed to be adequate, the generalized substitution system $\mathcal{SI}$ has model amalgamation.[9]

▶ **Theorem 20.** *For every institution $\mathcal{I}\colon \mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$ equipped with an adequate quantification space $\mathbb{Q}$ of representable signature extensions and with compatible categories $\mathbb{S}\mathrm{ubst}_\Sigma$ of $\mathbb{Q}$-representable Σ-substitutions, $\mathcal{SI}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$ is a generalized substitution system that has model amalgamation.* ◀

[9] This property is essential for ensuring that the satisfaction of clauses and queries is invariant under change of notation; in general, it means that, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and signature of Σ-variables X, we can amalgamate those models M' of Σ' and N of X that have the same Σ-reduct.

▶ **Example 21.** Both institutions QF-FOL$_=$ and QF-HNK, in combination with the extensions of signatures with constants and with the first-order and higher-order substitutions outlined in Example 4, give rise to generalized substitution systems.

5 Logic programming over an arbitrary institution

The view we take here is that the logic programming paradigm can be developed over an arbitrary institution $\mathcal{I}\colon \mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$ by considering logic-programming frameworks and languages as in [29] defined over the generalized substitution system $\mathcal{SI}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$ introduced in Section 4.3. To this end, we assume that $\mathcal{I}\colon \mathbb{S}\mathrm{ig} \to \mathbb{R}\mathrm{oom}$ is an institution that satisfies the hypotheses of Theorem 20, and we let $\mathcal{L}$ be a logic-programming language whose underlying generalized substitution system is derived from $\mathcal{I}$.[10]

Under the additional assumption that, for every signature Σ, the identity 1_Σ is a signature of variables – the 'empty' signature of Σ-variables – the general institution-independent versions of Herbrand's theorem presented in [5, 6] can be obtained as concrete instances of Herbrand's theorem for abstract logic-programming languages. In particular, the equivalence $1 \Leftrightarrow 2$ of Theorem 25 below captures the denotational aspect of the result – how the problem of checking whether a logic program entails a given query (formalized as an existential sentence) can be reduced from all models of the program to those that are initial; on the other hand, the equivalence $2 \Leftrightarrow 3$ emphasizes the operational aspect of the theorem – the correspondence between those expansions of the program's initial model that satisfy the underlying (quantifier-free) sentence of the query and the possible solutions to the query.

To start with, let us recall that, in any category, an object A is *projective* with respect to an arrow $e\colon B \to C$ provided that every other arrow $f\colon A \to C$ can be factored through e as $f = h \fatsemi e$, for some arrow $h\colon A \to B$. For instance, as a consequence of the axiom of choice, for every QF-FOL$_=$-signature extension $\langle S, F\rangle \subseteq \langle S, F \cup X\rangle$, the free algebra $\mathrm{T}_F(X)$ – that is the representation of the inclusion $\langle S, F\rangle \subseteq \langle S, F \cup X\rangle$ – is projective with respect to all epimorphisms, and in particular with respect to all quotient homomorphisms between the initial models $0_{\langle S,F\rangle}$ of the signature $\langle S, F\rangle$ and $0_{\langle S,F\rangle,\Gamma}$ of sets of $\langle S, F\rangle$-clauses Γ.

The concept of *basic sentence* (see [4], and also [27], where it was studied under the name of *ground positive elementary sentence*) captures the satisfaction of the conjunctions of atomic sentences that are usually involved in defining logic-programming queries.

▶ **Definition 22** (Basic sentence)**.** For any signature Σ, a sentence ρ is said to be *basic* if there exists a model M_ρ such that, for every Σ-model M, $M \models_\Sigma \rho$ if and only if there exists a model homomorphism $M_\rho \to M$.

▶ **Example 23.** In the institution QF-FOL$_=$, every (finite) conjunction of first-order equational atoms forms a basic sentence (see, for example, [6]). This property does not hold in general for QF-HNK, for which one can define higher-order equational atoms of the form $\sigma_1\, f = \sigma_2\, f$, with $f\colon s \to s$ and $\sigma_1, \sigma_2\colon (s \to s) \to s'$, that are not basic (see [2, 6]).

We also recall from [29] the following concept of reachability.

▶ **Definition 24** (Reachable model)**.** Given an extension $\chi\colon \Sigma \to \Sigma(\chi)$, a Σ-model M is *χ-reachable* if for every χ-expansion N of M there exists a substitution $\psi\colon \chi \to \chi'$ such that

[10] It should be noted that, in the present paper, we do not fully address the operational semantics of $\mathcal{L}$. A detailed presentation of the goal-directed rules – which, for first-order and higher-order equational logic programming, correspond to paramodulation – can be found in [29].

- χ' is conservative, in the sense that every Σ-model admits a χ'-expansion, and
- the canonical map $_\restriction_\chi : N \,/\, \mathrm{Mod}_\Sigma(\psi) \to M \,/\, \mathrm{Mod}(\chi')$ determined by the reduct functor $\mathrm{Mod}(\chi)$ is surjective on objects.

The above preliminaries enable us to recast the institution-independent versions of Herbrand's theorem in the context of abstract logic-programming languages.

▶ **Theorem 25.** *Consider a logic program* $\langle \Sigma, \Gamma \rangle$[11] *and a* Σ*-query* $\exists \chi \cdot \rho$ *such that*

- *both the signature* Σ *and the program* $\langle \Sigma, \Gamma \rangle$ *have initial models* 0_Σ *and* $0_{\Sigma,\Gamma}$*,*
- M_χ *is projective with respect to the unique homomorphism* $!_\Gamma : 0_\Sigma \to 0_{\Sigma,\Gamma}$*, and*
- *the* $\Sigma(\chi)$*-sentence* ρ *is basic.*

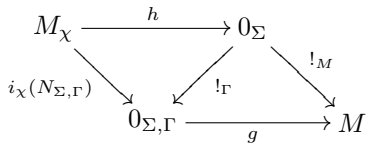
Then the following statements are equivalent:

1. $\Gamma \models_\Sigma \exists \chi \cdot \rho$.
2. $0_{\Sigma,\Gamma} \models_\Sigma \exists \chi \cdot \rho$.
3. *There exists a substitution* $\psi : \chi \to \chi'$ *such that* χ' *is conservative and* $\Gamma \models_\Sigma \forall \chi' \cdot \psi(\rho)$.

Proof. According to [29, Theorem 5.12], it suffices to prove that $0_{\Sigma,\Gamma}$ is χ-reachable and that ρ is preserved by χ-homomorphisms. The latter property follows from the assumption that ρ is basic (see [4]). Therefore, we will focus solely on proving that $0_{\Sigma,\Gamma}$ is χ-reachable.

Let $N_{\Sigma,\Gamma}$ be a χ-expansion of $0_{\Sigma,\Gamma}$. Since χ is a representable extension of signatures (by hypothesis), we know that $i_\chi(N_{\Sigma,\Gamma}) : M_\chi \to 0_{\Sigma,\Gamma}$ is an object of the comma category $M_\chi / \mathrm{Mod}(\Sigma)$; and because its representation, M_χ, is projective with respect to $!_\Gamma : 0_\Sigma \to 0_{\Sigma,\Gamma}$, it follows that there exists a homomorphism $h : M_\chi \to 0_\Sigma$ such that $h \fatsemi !_\Gamma = i_\chi(N_{\Sigma,\Gamma})$. Moreover, since the identity 1_Σ is a signature of Σ-variables – which, by hypothesis, is also representable – we deduce that 0_Σ is (isomorphic to) the representation M_{1_Σ} of 1_Σ. By the representability of Σ-substitutions, we further obtain the substitution $\left(R^{\mathbb{Q}}_{\Sigma'}\right)^{-1}(h) : \chi \to 1_\Sigma$, which we will henceforth denote by ψ. All we need to show now is that the canonical map $_\restriction_\Sigma : N_{\Sigma,\Gamma} \,/\, \mathrm{Mod}_\Sigma(\psi) \to 0_{\Sigma,\Gamma} \,/\, \mathrm{Mod}(\Sigma)$ is surjective on objects.

To this end, notice that every Σ-model homomorphism $g : 0_{\Sigma,\Gamma} \to M$ can be viewed as an arrow in $M_\chi \,/\, \mathrm{Mod}(\Sigma)$ between $i_\chi(N_{\Sigma,\Gamma})$ and $i_\chi(N_{\Sigma,\Gamma}) \fatsemi g$, from which we deduce that $i_\chi^{-1}(g)$ is a $\Sigma(\chi)$-model homomorphism between $N_{\Sigma,\Gamma}$ and $N = i_\chi^{-1}(i_\chi(N_{\Sigma,\Gamma}) \fatsemi g)$. In addition, by Proposition 14 and the commutativity of the diagram below, we obtain $M\restriction_\psi = i_\chi^{-1}(h \fatsemi !_M) = i_\chi^{-1}(i_\chi(N_{\Sigma,\Gamma}) \fatsemi g) = N$, thus confirming that $i_\chi^{-1}(g) : N_{\Sigma,\Gamma} \to M\restriction_\psi$ is an object of $N_{\Sigma,\Gamma} \,/\, \mathrm{Mod}_\Sigma(\psi)$. The conclusion of the theorem follows by observing that $N\restriction_\Sigma = |i_\chi(N_{\Sigma,\Gamma}) \fatsemi g|_{M_\chi} = M$ and $i_\chi^{-1}(g)\restriction_\Sigma = |g|_{M_\chi} = g$.



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[11] For simplicity, we only consider here logic programs defined as theory presentations. The same result can still be stated for more complex, structured logic programs as in [29].

6 Conclusions

In this paper, we have examined the connection between the institution-independent approach to Herbrand's theorem reported in [5] and the abstract axiomatic theory of logic programming that we previously proposed in [29]. We have first shown that, for an arbitrary but fixed signature Σ of an institution $\mathcal{I}$, any class of $\mathcal{I}$-signature morphisms gives rise to a canonical Σ-substitution system whose substitutions correspond precisely to the institution-independent concept of substitution. Lifting this result to institutions and generalized substitution systems – so as to enable the application of the general variant of Herbrand's theorem from [29] – proved to be much more difficult, and it required the development of a number of new properties and results concerning quantification spaces and the representability of signature morphisms. To summarize, we have determined that any institution equipped with an adequate quantification space of representable signature extensions and with compatible categories of representable substitutions leads to a generalized substitution system. Moreover, we showed that the resulting generalized substitution system has model amalgamation, and thus that it forms a suitable foundation for defining logic-programming languages.

The most problematic aspect of the derivation of a generalized substitution system is the translation of institution-independent substitutions along signature morphisms, for which one still has to check properties such as compatibility for each particular institution of interest. For this reason, a promising line of research would be to explore alternative, more syntactic, and also more specific notions of substitution, inspired for example by the recent study [21] on derived signature morphisms and substitutions in the context of institutional monads.

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