Extensions of functors from Set to $\mathscr{V}$-cat*

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Abstract

We show that for a commutative quantale $\mathscr{V}$ every functor $\mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ has an enriched left-Kan extension. As a consequence, coalgebras over Set are subsumed by coalgebras over $\mathscr{V}\text{-}\mathsf{cat}$. Moreover, one can build functors on $\mathscr{V}\text{-}\mathsf{cat}$ by equipping Set-functors with a metric.

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1 Introduction

Coalgebras for a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ capture a wide variety of dynamic systems [18]. Moreover, the category $\mathsf{Coalg}(T)$ of coalgebras has a rich structure, which dualizes to some extent the theory of universal algebra. For example, an important role is played by final (or cofree) coalgebras, which give rise to a notion of behavioural equivalence and coinduction. One says that two elements of two coalgebras are behaviourally equivalent (or bisimilar), if they are identified by the morphisms into the final coalgebra. The coinduction principle states that on the final coalgebra two bisimilar elements are equal.

Rutten [17] and Worrell [20, 21] investigate how to account for richer notions of behaviour. For example, we might want to say that one behaviour is smaller than (or, is simulated by) another behaviour. Or we might want to measure distances between behaviours by real numbers. As proposed by Rutten [17], the right framework to develop a theory of metric coalgebras that parallels the theory of coalgebras over Set is given by coalgebras over $\mathscr{V}\text{-}\mathsf{cat}$, in the sense we are going to explain now.

It was Lawvere [14] who discovered that metric spaces are categories enriched over the category

$$((0, \infty], \geq_{\mathbb{R}}), +, 0).$$

That an enriched category $\mathscr{X}$ with homs $\mathscr{X}(x, y) \in [0, \infty]$ has identities means $0 = \mathscr{X}(x, x)$ and composition becomes the triangle inequality $\mathscr{X}(x, y) + \mathscr{X}(y, z) \geq_{\mathbb{R}} \mathscr{X}(x, z)$. Thus, enriched categories are nothing but generalized metric spaces, generalized in the sense that distances need not be symmetric and that $\mathscr{X}(x, y) = \mathscr{X}(y, x) = 0$ is not equality but merely

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an equivalence relation. This interpretation of enriched categories is meaningful not only for $\mathscr{V} = (([0,\infty], \geq_{\mathbb{R}}), +, 0)$, but for any *commutative quantale* $\mathscr{V}$. A category enriched over $\mathscr{V}$ is then called a $\mathscr{V}$-category.

For a detailed discussion of examples showing the relevance of this approach to the denotational semantics of programmming languages we refer to Worrell [21, Chapter 4].

In this paper, we contribute a theorem about the category $\mathscr{V}$-cat of categories enriched over a commutative quantale $\mathscr{V}$. The theorem states that any functor $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ has an enriched left Kan extension along the 'discrete' functor $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$. Moreover, the proof of the theorem shows how to compute the Kan extension $H^\sharp$ on a $\mathscr{V}$-category $\mathscr{X}$ by applying H to the '$\mathscr{V}$-nerve' of $\mathscr{X}$ and then taking an appropriate colimit in $\mathscr{V}$-cat. For example, the extension of $D^{\mathscr{V}}P : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$, where $P : \mathsf{Set} \longrightarrow \mathsf{Set}$ is the powerset functor, yields the familiar Pompeiu-Hausdorff metric, if the quantale is assumed to be constructively completely distributive.

Apart from allowing us to construct functors on $\mathscr{V}$-cat, the theorem also allows us to establish that for any commutative quantale $\mathscr{V}$ (satisfying some mild properties) the setting of coalgebras enriched over $\mathscr{V}$-cat is indeed richer than the setting of Set-coalgebras in the following sense. For any functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ we can define its $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ to be the left Kan extension of $D^{\mathscr{V}}T$ along $D^{\mathscr{V}}$. Then there is a functor $\widetilde{D}^{\mathscr{V}} : \mathsf{Coalg}(T) \longrightarrow \mathsf{Coalg}(T_{\mathscr{V}})$ which is *right* adjoint and therefore preserves behaviours. In other words, in the world of $\mathscr{V}$-categories all functors $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ are still available via their $\mathscr{V}$-cat-ifications. On the other hand, it happens often for an endofunctor T on Set to carry an interesting $\mathscr{V}$-metric, which in turn determines a lifting $\overline{T}$ of T to $\mathscr{V}$-cat. In such case the discrete $\mathscr{V}$-cat-functor has as *ordinary* right adjoint the forgetful functor $\widetilde{V}^{\mathscr{V}} : \mathsf{Coalg}(\overline{T}) \longrightarrow \mathsf{Coalg}(T)$, which consequently preserves behaviors.

2 Preliminaries

In this section we gather all the necessary technicalities and notation from category theory enriched in a complete and cocomplete symmetric monoidal category that we shall use later. For the standard notions of enriched categories, enriched functors and enriched natural transformations we refer to Kelly's book [12].

We shall mainly use two prominent enrichments: that in a quantale $\mathscr{V}$ and that in the category $\mathscr{V}$-cat of *small* $\mathscr{V}$-categories and $\mathscr{V}$-functors for a quantale $\mathscr{V}$. We spell out in more details how the relevant notions look like, and carefully write all the enrichment-prefixes. In particular, the underlying category of an enriched category will be denoted by the same symbol, followed by the subscript "o" as usual.

2.1 Categories and functors enriched in a quantale

Suppose $\mathscr{V} = (\mathscr{V}_o, \otimes, e, [-,-])$ is a *quantale*. More in detail: $\mathscr{V}_o$ is a complete lattice, equipped with the commutative and associative monotone binary operation $\otimes$, called the *tensor*. We require the element e to be a *unit* of tensor. Furthermore, we require every monotone map $- \otimes r : \mathscr{V}_o \longrightarrow \mathscr{V}_o$ to have a right adjoint $[r, -] : \mathscr{V}_o \longrightarrow \mathscr{V}_o$. We call $[-,-]$ the *internal hom* of $\mathscr{V}_o$.

Quantales are the "simplest" complete and cocomplete symmetric monoidal closed categories. Therefore, one can define $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations. Before we say what these are, let us mention several examples of quantales.

▶ **Examples 2.1.**

1. The two-element chain $\mathbb{2} = \{0, 1\}$ with the usual order, and tensor $r \otimes s = r \wedge s$.
2. The real half line $([0, \infty], \geq_{\mathbb{R}})$, with (extended) addition as tensor product.
3. The unit interval $([0, 1], \geq_{\mathbb{R}})$ with tensor product $r \otimes s = \max(r, s)$.
4. The poset of all monotone functions $f : [0, \infty] \longrightarrow [0, 1]$ such that the equality $f(x) = \bigvee_{y<x} f(y)$ holds, with the pointwise order. It becomes a quantale with the tensor product
$$f \otimes g(z) = \bigvee_{x+y \leq z} f(x) \cdot g(y)$$
having as unit the function mapping all nonzero elements to 1, and 0 to itself [10].
5. The three-element chain $\mathbb{3} = \{0, 1, 2\}$ with usual order, and the (unique!) commutative tensor product with unit 1, which necessarily satisfies $2 \otimes 2 = 2$ (which can be seen by tensoring both sides of $1 \leq 2$ with 2). ◀

A (*small*) $\mathscr{V}$*-category* $\mathscr{X}$ consists of a (small) set of objects, together with an object $\mathscr{X}(x', x)$ in $\mathscr{V}_o$ for each pair x', x of objects, subject to the following axioms
$$e \leq \mathscr{X}(x, x), \quad \mathscr{X}(x', x) \otimes \mathscr{X}(x'', x') \leq \mathscr{X}(x'', x)$$
for all objects x'', x' and x in $\mathscr{X}$. A $\mathscr{V}$-category $\mathscr{X}$ is called *discrete* if $\mathscr{X}(x', x) = e$ for $x' = x$, and $\bot$ otherwise.

A $\mathscr{V}$*-functor* $f : \mathscr{X} \longrightarrow \mathscr{Y}$ is given by the object-assignment $x \mapsto fx$, such that
$$\mathscr{X}(x', x) \leq \mathscr{Y}(fx', fx)$$
holds for all x', x.

A $\mathscr{V}$*-natural transformation* $f \longrightarrow g$ is given whenever
$$e \leq \mathscr{Y}(fx, gx)$$
holds for all x. Thus, there is at most one $\mathscr{V}$-natural transformation between f and g.

▶ **Example 2.2.** The two-element chain $\mathbb{2}$ is a quantale. A small $\mathbb{2}$-category[1] $\mathscr{X}$ is precisely a *preorder*, where $x' \leq x$ iff $\mathscr{X}(x', x) = 1$, while a $\mathbb{2}$-functor $f : \mathscr{X} \longrightarrow \mathscr{Y}$ is a monotone map. A $\mathbb{2}$-natural transformation $f \to g$ expresses that $fx \leq gx$ holds for every x. Thus $\mathbb{2}$-cat is the category Preord of preorders and monotone maps.

A good intution is that $\mathscr{V}$-categories are (rather general) metric spaces and $\mathscr{V}$-functors are nonexpanding maps. This intuition goes back to Lawvere [14]. We show next some examples that explain this intuition. For more details, see also [16].

▶ **Examples 2.3.**

1. Let $\mathscr{V}$ be the real half line $([0, \infty], \geq_{\mathbb{R}}, +, 0)$ as in Example 2.1.2. It is easy to see that a small $\mathscr{V}$-category can be identified with a set X and a mapping $d_X : X \times X \longrightarrow [0, \infty]$ such that $\langle X, d_X \rangle$ is a *generalized metric space*. The slight generalization of the usual notion lies in the fact that the distance function d is not necessarily symmetric and $d_X(x', x) = 0$ does not necessarily entail $x' = x$.
   A $\mathscr{V}$-functor $f : (X, d_X) \longrightarrow (Y, d_Y)$ is then a exactly a *nonexpanding mapping*, i.e., one satisfying the inequality $d_Y(fx', fx) \leq d_X(x', x)$ for every $x, x' \in X$.
   The existence of a $\mathscr{V}$-natural transformation $f \longrightarrow g$ means that $\bigvee_x d_Y(fx, gx) = 0$, i.e., the distance $d_Y(fx, gx)$ is 0, for every $x \in X$.
2. For the unit interval $\mathscr{V} = ([0, 1], \geq_{\mathbb{R}}, \max, 0)$ from Example 2.1.3, a $\mathscr{V}$-category is a *generalized ultrametric space* $\langle X, d_X : X \times X \longrightarrow [0, 1] \rangle$ [16, 20]. Again, the slight generalization of the usual notion lies in the fact that the distance function d is not necessarily

[1] To not be confounded with the notion of a 2-category, that is, a Cat-enriched category.

symmetric and $d_X(x', x) = 0$ does not necessarily entail $x = x'$. Similarly, $\mathscr{V}$-functors are precisely the nonexpanding maps, and the existence of a $\mathscr{V}$-natural transformation $f \longrightarrow g : \langle X, d_X\rangle \longrightarrow \langle Y, d_Y\rangle$ means, again, that $\bigvee_x d_Y(fx, gx) = 0$, i.e., the distance $d_Y(fx, gx)$ is 0, for every $x \in X$.

3. Using the quantale $\mathscr{V}$ from Example 2.1.4 leads to *probabilistic metric spaces*: for a $\mathscr{V}$-category $\mathscr{X}$, and for every pair x, x' of objects of $\mathscr{X}$, the hom-object is a function $\mathscr{X}(x', x) : [0, \infty] \longrightarrow [0, 1]$ with the intuitive meaning $\mathscr{X}(x', x)(r) = s$ holds iff s is the probability that the distance from x' to x is smaller than r. See [6, 10].
4. Finally, for the three-element quantale from Example 2.1.5, $\mathscr{V}$-enriched categories arose in the model of concurrency proposed by Gaifman and Pratt [8] under the name of *prossets*. Explicitly, the objects of a $\mathscr{V}$-category can be seen as events subject to a schedule, endowed with a preorder $\leq$ and a binary relation $\prec$, where $x \leq y$ iff $\mathscr{X}(x, y) \geq 1$ (with the interpretation that "y cannot begin before x begins, and cannot complete before x completes"), and $x \prec y$ iff $\mathscr{X}(x, y) = 2$ (which is intended to mean "y cannot begin until x has completed").

2.2 Categories, functors and natural transformations, enriched in $\mathscr{V}$-cat

Suppose that $\mathscr{V} = (\mathscr{V}_o, \otimes, e, [-,-])$ is a quantale. We denote by $\mathscr{V}$-cat_o the *ordinary* category of all small $\mathscr{V}$-categories and all $\mathscr{V}$-functors between them.

We recall (see for example [21]) that the ordinary category $\mathscr{V}$-cat_o has a monoidal closed structure. The *tensor product* $\mathscr{X} \otimes \mathscr{Y}$ is inherited from $\mathscr{V}$. Namely, $\mathscr{X} \otimes \mathscr{Y}$ has as objects the corresponding pairs of objects and we put

$$(\mathscr{X} \otimes \mathscr{Y})((x', y'), (x, y)) = \mathscr{X}(x', x) \otimes \mathscr{Y}(y', y)$$

The *unit* for the tensor product is the $\mathscr{V}$-category $\mathbb{1}$, with one object 0 and $\mathscr{V}$-hom $\mathbb{1}(0, 0) = e$.

The $\mathscr{V}$-functor $- \otimes \mathscr{Y} : \mathscr{V}$-$\mathsf{cat}_o \longrightarrow \mathscr{V}$-$\mathsf{cat}_o$ has a right adjoint $[\mathscr{Y}, -]$. Explicitly, $[\mathscr{Y}, \mathscr{Z}]$ is the following $\mathscr{V}$-category:

1. Objects of $[\mathscr{Y}, \mathscr{Z}]$ are $\mathscr{V}$-functors from $\mathscr{Y}$ to $\mathscr{Z}$.
2. The "distance" $[\mathscr{Y}, \mathscr{Z}](f, g)$ is $\bigwedge_y \mathscr{Z}(fy, gy)$.

It follows from [13] that the symmetric monoidal closed category $(\mathscr{V}$-$\mathsf{cat}_o, \otimes, \mathbb{1}, [-,-])$ is complete and cocomplete, with generator consisting of $\mathscr{V}$-categories of the form 2_r, $r \in \mathscr{V}_o$. Here, every 2_r has two objects 0 and 1, with $\mathscr{V}$-homs

$$2_r(0, 0) = 2_r(1, 1) = e\,,\ 2_r(0, 1) = r\,,\ 2_r(1, 0) = \bot \tag{1}$$

Thus we can define $\mathscr{V}$-cat-enriched categories, $\mathscr{V}$-cat-functors and $\mathscr{V}$-cat-natural transformations.

A (*small*) $\mathscr{V}$-cat-*category* $\mathbb{X}$ consists of a (small) set of objects X, Y, Z, ..., a small $\mathscr{V}$-category $\mathbb{X}(X, Y)$ for every pair X, Y of objects, and $\mathscr{V}$-functors

$$u_X : \mathbb{1} \longrightarrow \mathbb{X}(X, X), \quad c_{X,Y,Z} : \mathbb{X}(Y, Z) \otimes \mathbb{X}(X, Y) \longrightarrow \mathbb{X}(X, Z)$$

that represent the identity and composition and satisfy the usual axioms [12]:

$$\begin{array}{ccc}
\mathbb{X}(Z,W) \otimes \mathbb{X}(Y,Z) \otimes \mathbb{X}(X,Y) & \xrightarrow{1 \otimes c_{X,Y,Z}} & \mathbb{X}(Z,W) \otimes \mathbb{X}(X,Z) \\
\downarrow{\scriptstyle c_{Y,Z,W} \otimes 1} & & \downarrow{\scriptstyle c_{X,Z,W}} \\
\mathbb{X}(Y,W) \otimes \mathbb{X}(X,Y) & \xrightarrow{c_{X,Y,W}} & \mathbb{X}(X,W)
\end{array}$$

$$\begin{array}{ccc}
\mathbb{1} \otimes \mathbb{X}(X,Y) & \xrightarrow{u_Y \otimes 1} & \mathbb{X}(Y,Y) \otimes \mathbb{X}(X,Y) \\
& \searrow^{\cong} & \downarrow{\scriptstyle c_{X,Y,Y}} \\
& & \mathbb{X}(X,Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{X}(X,Y) \otimes \mathbb{X}(X,X) & \xleftarrow{1 \otimes u_X} & \mathbb{X}(X,Y) \otimes \mathbb{1} \\
\downarrow{\scriptstyle c_{X,X,Y}} & \swarrow^{\cong} & \\
\mathbb{X}(X,Y) & &
\end{array}$$

Objects of $\mathbb{X}(X,Y)$ will be sometimes denoted by $f : X \longrightarrow Y$ and their "distance" by $\mathbb{X}(X,Y)(f,g)$ in $\mathscr{V}$. The action of $c_{X,Y,Z}$ at objects (f',f) in $\mathbb{X}(Y,Z) \otimes \mathbb{X}(X,Y)$ is denoted simply by $f' \cdot f$, and for their distances the inequality below (expressing that $c_{X,Y,Z}$ is a $\mathscr{V}$-functor) holds:

$$(\mathbb{X}(Y,Z) \otimes \mathbb{X}(X,Y))\,((f',g'),(f,g)) \leq \mathbb{X}(X,Z)(f' \cdot f, g' \cdot g)$$

A $\mathscr{V}$-cat-*functor* $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is given by:

1. The assignment $X \mapsto FX$ on objects.
2. For each pair of objects X, X' in $\mathbb{X}$, a $\mathscr{V}$-functor $F_{X',X} : \mathbb{X}(X',X) \longrightarrow \mathbb{Y}(FX',FX)$, whose action on objects $f : X' \longrightarrow X$ is denoted by $Ff : FX' \longrightarrow FX$. For the distances we have the inequality
$$\mathbb{X}(X',X)(f',f) \leq \mathbb{Y}(FX',FX)(Ff',Ff)$$

Of course, the diagrams of $\mathscr{V}$-functors below, expressing the preservation of unit and composition, should commute:

$$\begin{array}{ccc} \mathbb{X}(X,X) & \xrightarrow{F_{X,X}} & \mathbb{Y}(FX,FX) \\ & {\scriptstyle u_X}\nwarrow \quad \nearrow{\scriptstyle u_{FX}} & \\ & \mathbb{1} & \end{array} \qquad \begin{array}{ccc} \mathbb{X}(Y,Z) \otimes \mathbb{X}(X,Y) & \xrightarrow{F_{Y,Z} \otimes F_{X,Y}} & \mathbb{Y}(FY,FZ) \otimes \mathbb{Y}(FX,FY) \\ {\scriptstyle c_{X,Y,Z}}\downarrow & & \downarrow{\scriptstyle c_{X,Y,Z}} \\ \mathbb{X}(X,Z) & \xrightarrow[F_{X,Z}]{} & \mathbb{Y}(FX,FZ) \end{array}$$

Given $F, G : \mathbb{X} \longrightarrow \mathbb{Y}$, a $\mathscr{V}$-cat-natural transformation $\tau : F \longrightarrow G$ is given by a collection of $\mathscr{V}$-cat-functors $\tau_X : \mathbb{1} \longrightarrow \mathbb{Y}(FX,GX)$, such that the diagram

$$\begin{array}{ccccc} & & \mathbb{1} \otimes \mathbb{X}(X',X) & \xrightarrow{\tau_X \otimes F_{X',X}} & \mathbb{Y}(FX,GX) \otimes \mathbb{Y}(FX',FX) \\ & \overset{\cong}{\nearrow} & & & \searrow^{c_{FX',FX,GX}} \\ \mathbb{X}(X',X) & & & & \mathbb{Y}(FX',GX) \\ & \underset{\cong}{\searrow} & & & \nearrow_{c_{FX',GX',GX}} \\ & & \mathbb{X}(X',X) \otimes \mathbb{1} & \xrightarrow[G_{X',X} \otimes \tau_{X'}]{} & \mathbb{Y}(FX',GX) \otimes \mathbb{Y}(FX',GX') \end{array}$$

of $\mathscr{V}$-functors commutes. We shall abuse the notation and denote by $\tau_X : FX \longrightarrow GX$ the image in $\mathbb{Y}(FX,GX)$ of 0 in $\mathbb{1}$ under $\tau_X : \mathbb{1} \longrightarrow \mathbb{Y}(FX,GX)$. The above diagram (when read at the object-assignments of the ambient $\mathscr{V}$-functors) then translates as the equality

$$Gf \cdot \tau_{X'} = \tau_X \cdot Ff$$

of objects of the $\mathscr{V}$-category $\mathbb{Y}(FX',GX)$, for every object $f : X' \longrightarrow X$. On hom-objects, the above diagram says nothing[2] (recall that $\mathscr{V}_o$ is a poset, hence there are no parallel pairs of morphisms in $\mathscr{V}_o$).

Since $\mathscr{V}$-categories are "generalized metric spaces" (as seen in Examples 2.3), $\mathscr{V}$-cat-categories are "locally" metric spaces and $\mathscr{V}$-cat-functors are "locally" nonexpanding.

The last bit of notation standard from enriched category theory concerns colimits. We introduce it for $\mathscr{V}$-cat-categories.

▶ **Definition 2.4.** A *colimit* of a diagram $D : \mathbb{D} \longrightarrow \mathbb{X}$ weighted by a $\mathscr{V}$-cat-functor $\varphi : \mathbb{D}^{op} \longrightarrow \mathscr{V}$-cat consists of an object $\varphi * D$ of $\mathbb{X}$, together with an isomorphism

$$\mathbb{X}(\varphi * D, X) \cong [\mathbb{D}^{op}, \mathscr{V}\text{-cat}](\varphi, \mathbb{X}(D-,X))$$

which is $\mathscr{V}$-cat-natural in X.

In case $\mathbb{D}$ is the one-object $\mathscr{V}$-cat-category, we can identify the $\mathscr{V}$-cat-functor D with an object P of $\mathbb{X}$ and φ with a $\mathscr{V}$-category $\mathscr{C}$. We write then $\mathscr{C} \bullet P$ instead of $\varphi * D$.

[2] This is well-known for Preord-natural transformations: one only needs to verify ordinary naturality.

▶ **Example 2.5.** Let Set denote in the sequel the free $\mathscr{V}$-cat-category on the ordinary category of sets and functions Set_o. This means that $\mathsf{Set}(X', X) = \mathsf{Set}_o(X', X) \bullet \mathbb{1}$, hence the homs of Set are copowers of the one-element "metric" space, indexed by set-theoretical maps from X' to X (that is, $\mathsf{Set}(X', X)$ is a discrete $\mathscr{V}$-category). Observe that ordinary functors $\mathsf{Set}_o \longrightarrow \mathsf{Set}_o$ automatically induce $\mathscr{V}$-cat-enriched functors $\mathsf{Set} \longrightarrow \mathsf{Set}$, and similarly for natural transformations between such ordinary functors.

3 Extensions from Set to $\mathscr{V}$-cat

From now on, we fix a quantale $\mathscr{V}$. We consider $\mathscr{V}$-cat enriched over itself as usual, using its internal hom described in Section 2.2, and Set as free $\mathscr{V}$-cat-category (Example 2.5). Denote by $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ the corresponding $\mathscr{V}$-cat-enriched embedding. Explicitly, $D^{\mathscr{V}}$ maps a set X to the discrete $\mathscr{V}$-category having X as set of objects.
Notice that there is an *ordinary* adjunction $D_o^{\mathscr{V}} \dashv V^{\mathscr{V}} : \mathscr{V}\text{-}\mathsf{cat}_o \longrightarrow \mathsf{Set}_o$ where the (ordinary) functor $V^{\mathscr{V}}$ maps a $\mathscr{V}$-category $\mathscr{X}$ to its set of objects of $\mathscr{X}$.

▶ **Definition 3.1.** Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, $\overline{T} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ be $\mathscr{V}$-cat-functors.

- We say that a $\mathscr{V}$-cat-natural isomorphism

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathsf{cat} & \xrightarrow{\overline{T}} & \mathscr{V}\text{-}\mathsf{cat} \\ {\scriptstyle D^{\mathscr{V}}}\uparrow & \nwarrow \alpha & \uparrow{\scriptstyle D^{\mathscr{V}}} \\ \mathsf{Set} & \xrightarrow[T]{} & \mathsf{Set} \end{array}$$

  of $\mathscr{V}$-cat-functors exhibits $\overline{T}$ as an *extension* of T. If additionally the above isomorphism α is the unit of a left Kan extension, i.e., if $\overline{T} = \mathrm{Lan}_{D^{\mathscr{V}}}(D^{\mathscr{V}}T)$ holds, then we say that α exhibits $\overline{T}$ as the $\mathscr{V}$-cat-*ification* of T, and we shall denote it by $T_{\mathscr{V}}$.
- We say that a natural isomorphism

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathsf{cat}_o & \xrightarrow{\overline{T}_o} & \mathscr{V}\text{-}\mathsf{cat}_o \\ {\scriptstyle V^{\mathscr{V}}}\downarrow & \nwarrow \beta & \downarrow{\scriptstyle V^{\mathscr{V}}} \\ \mathsf{Set}_o & \xrightarrow[T_o]{} & \mathsf{Set}_o \end{array}$$

  of ordinary functors exhibits $\overline{T}$ as a *lifting* of T.

▶ **Examples 3.2.**

1. The identity $\mathscr{V}$-cat-functor $\mathsf{Id} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is *always* an extension and a lifting of the identity ($\mathscr{V}$-cat-)functor on Set.
   In case the quantale has an element r satisfying $e \leq r$ and $r \otimes r \leq r$ (consequently, $r \otimes r = r$), then the identity on Set has another lifting, namely $\mathsf{Id}_r : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$, mapping a $\mathscr{V}$-category $\mathscr{X}$ to the $\mathscr{V}$-category with same objects, and $\mathscr{V}$-homs $(\mathsf{Id}_r \mathscr{X})(x', x) = \mathscr{X}(x', x) \otimes r$ "shrinked" by r, and acting as identity on $\mathscr{V}$-functors.
2. Extensions and liftings need not be unique. We have seen above an example for liftings, now we give one for extensions. Suppose $\mathscr{V} = \mathbb{2}$ (thus $\mathscr{V}$-cat is Preord). We shall then denote simply by $D : \mathsf{Set} \longrightarrow \mathsf{Preord}$ the discrete functor, omitting the superscript $\mathbb{2}$. It has as ($\mathbb{2}$-enriched!) left adjoint the functor $C : \mathsf{Preord} \longrightarrow \mathsf{Set}$ assigning to any preorder X the set of its connected components. The composite $\pi = DC : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ is an extension of $\mathsf{Id} : \mathsf{Set} \longrightarrow \mathsf{Set}$. The latter follows from the fact that $\pi D \cong DCD \cong D$ holds by virtue of the counit of $C \dashv D$. Hence both Id and π are extensions of $\mathsf{Id} : \mathsf{Set} \longrightarrow \mathsf{Set}$.
   We shall later show (Examples 3.7) that $\mathsf{Id} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is, in fact, a $\mathscr{V}$-cat-ification of the identity functor on Set, for an *arbitrary* quantale $\mathscr{V}$.

3. A $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ exists for every *accessible* functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ for rather trivial reasons. More in detail, if T is λ-accessible for a regular cardinal, then $T = \mathrm{Lan}_{J_\lambda}(TJ_\lambda)$, where $J_\lambda : \mathsf{Set}_\lambda \longrightarrow \mathsf{Set}$ is the inclusion of the full subcategory Set_λ spanned by λ-small sets. Consequently,
$$T_{\mathscr{V}} = \mathrm{Lan}_{D^{\mathscr{V}} J_\lambda}(D^{\mathscr{V}} T J_\lambda)$$
exhibits $T_{\mathscr{V}}$ as $\mathrm{Lan}_{D^{\mathscr{V}}}(D^{\mathscr{V}} T)$ by [12, Theorem 4.47]. In particular, the $\mathscr{V}$-cat-ification $(T_\Sigma)_{\mathscr{V}}$ exists for every polynomial functor
$$T_\Sigma X = \coprod_n \mathsf{Set}(n, X) \bullet \Sigma n$$
where $\Sigma : |\mathsf{Set}_\lambda| \longrightarrow \mathsf{Set}$ is a λ-ary signature. We shall give an explicit formula for the $\mathscr{V}$-cat-ification $(T_\Sigma)_{\mathscr{V}}$ later. ◀

We plan to show that for each endofunctor T on Set, its $\mathscr{V}$-cat-ification exists. We shall obtain this from the more general result below, which also will provide examples of liftings.

▶ **Theorem 3.3.** *Every functor $H : \mathsf{Set} \longrightarrow \mathscr{V}$-cat has a $\mathscr{V}$-cat-enriched left Kan extension $H^\sharp : \mathscr{V}$-cat $\longrightarrow \mathscr{V}$-cat along $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}$-cat.*

Proof. We first introduce a $\mathscr{V}$-cat-functor $N : \mathbb{N}^{op} \longrightarrow \mathscr{V}$-cat. Its domain $\mathbb{N}$ is the *free* $\mathscr{V}$-cat-category built upon the following ordinary category N: the objects are all r in $\mathscr{V}_o$, together with an extra symbol Ω, with arrows $\delta_0^r : r \longrightarrow \Omega$ and $\delta_1^r : r \longrightarrow \Omega$, for all r in $\mathscr{V}_o$.

We define N to be the $\mathscr{V}$-cat-functor sending Ω to $\mathbb{1}$, and r to $\mathbb{2}_r$. Recall that $\mathbb{1}$ is the unit one-object $\mathscr{V}$-category with $\mathbb{1}(0,0) = e$, and $\mathbb{2}_r$ is the $\mathscr{V}$-category on two objects 0 and 1, with the only non-trivial "distance" $\mathbb{2}_r(0,1) = r$, as introduced in Equation (1). The action of N on arrows is defined as follows: $N\delta_0^r : \mathbb{1} \longrightarrow \mathbb{2}_r$ sends 0 to 0, while $N\delta_1^r : \mathbb{1} \longrightarrow \mathbb{2}_r$ sends 0 to 1.

Then, for every $\mathscr{V}$-category $\mathscr{X}$, we consider the following $\mathscr{V}$-cat-functor $D_{\mathscr{X}} : \mathbb{N} \longrightarrow \mathsf{Set}$. Since $\mathbb{N}$ is a free $\mathscr{V}$-cat-category, it suffices to define an ordinary functor $\mathsf{N} \longrightarrow \mathsf{Set}_o$. We put $D_{\mathscr{X}}\Omega$ to be the set of objects of $\mathscr{X}$. Every r is sent to the set $D_{\mathscr{X}} r$ of pairs (x', x) of objects such that $r \leq \mathscr{X}(x', x)$ holds. The mapping $D_{\mathscr{X}}\delta_0^r$ sends (x', x) to x' and $D_{\mathscr{X}}\delta_1^r$ sends (x', x) to x.

We prove the following facts:

1. The colimit $N * (D^{\mathscr{V}} D_{\mathscr{X}})$ in $\mathscr{V}$-cat is isomorphic to $\mathscr{X}$.
2. If we define $H^\sharp\mathscr{X}$ as the colimit $N * (HD_{\mathscr{X}})$, then the assignment $\mathscr{X} \mapsto H^\sharp\mathscr{X}$ can be extended to a $\mathscr{V}$-cat-functor that is a left Kan extension of H along $D^{\mathscr{V}}$.

Let us proceed:

1. The colimit $N * (D^{\mathscr{V}} D_{\mathscr{X}})$ exists in $\mathscr{V}$-cat, since the $\mathscr{V}$-cat-category $\mathbb{N}$ is small.
   To ease the notation, we put $D^{\mathscr{V}} D_{\mathscr{X}}\Omega = \mathscr{X}_\Omega$, $D^{\mathscr{V}} D_{\mathscr{X}} r = \mathscr{X}_r$, $D^{\mathscr{V}} D_{\mathscr{X}}\delta_0^r = \partial_0^r$, and $DD_{\mathscr{X}}\delta_1^r = \partial_1^r$.
   Let us analyze the defining isomorphism
   $$\mathscr{V}\text{-cat}(N * (D^{\mathscr{V}} D_{\mathscr{X}}), \mathscr{Y}) \cong [\mathbb{N}^{op}, \mathscr{V}\text{-cat}](N, \mathscr{V}\text{-cat}(D^{\mathscr{V}} D_{\mathscr{X}} -, \mathscr{Y}))$$
   of $\mathscr{V}$-categories, natural in $\mathscr{Y}$.
   The $\mathscr{V}$-category $[\mathbb{N}^{op}, \mathscr{V}\text{-cat}](N, \mathscr{V}\text{-cat}(D^{\mathscr{V}} D_{\mathscr{X}} -, \mathscr{Y}))$ of N-weighted "cocones" for $D^{\mathscr{V}} D_{\mathscr{X}}$ is described as follows:
   a. The objects are $\mathscr{V}$-cat-natural transformations $\tau : N \longrightarrow \mathscr{V}\text{-cat}(D^{\mathscr{V}} D_{\mathscr{X}} -, \mathscr{Y})$. Each such τ consists of $\mathscr{V}$-functors
      i. $\tau_\Omega : N\Omega \longrightarrow \mathscr{V}\text{-cat}(\mathscr{X}_\Omega, \mathscr{Y})$. Since $N\Omega = \mathbb{1}$, τ_Ω picks up a $\mathscr{V}$-functor $f_\Omega : \mathscr{X}_\Omega \longrightarrow \mathscr{Y}$. No other restrictions are imposed since $\mathbb{1}(0,0) = e$.

ii. $\tau_r : Nr \longrightarrow \mathscr{V}\text{-}\mathsf{cat}(\mathscr{X}_r, \mathscr{Y})$. This $\mathscr{V}$-functor picks up two $\mathscr{V}$-functors $f_0^r : \mathscr{X}_r \longrightarrow \mathscr{Y}$ and $f_1^r : \mathscr{X}_r \longrightarrow \mathscr{Y}$. Since $\mathscr{X}_r$ is discrete, both f_0 and f_1 are defined by their object-assignments only. There is, however, the constraint below, because $Nr = 2_r$:

$$r \leq \bigwedge_{r \leq \mathscr{X}(x',x)} \mathscr{Y}(f_0^r(x',x), f_1^r(x',x))$$

In addition to the above, there are various commutativity conditions since τ is natural. Explicitly, for $\delta_0^r : r \longrightarrow \Omega$, we have the commutative square

$$\begin{array}{ccc} N\Omega & \xrightarrow{\tau_\Omega} & \mathscr{V}\text{-}\mathsf{cat}(\mathscr{X}_\Omega, \mathscr{Y}) \\ {\scriptstyle N\delta_0^r}\downarrow & & \downarrow{\scriptstyle \mathscr{V}\text{-}\mathsf{cat}(\partial_0^r, \mathscr{Y})} \\ Nr & \xrightarrow[\tau_r]{} & \mathscr{V}\text{-}\mathsf{cat}(\mathscr{X}_r, \mathscr{Y}) \end{array}$$

that, on the level of objects, is the requirement $f_\Omega \cdot \partial_0^r = f_0^r$. Analogously, the requirement $f_\Omega \cdot \partial_1^r = f_1^r$ holds.

We conclude that to give τ reduces to a $\mathscr{V}$-functor $f_\Omega : \mathscr{X}_\Omega \longrightarrow \mathscr{Y}$ (and, recall, this $\mathscr{V}$-functor is given just by the object-assignment $x \mapsto f_\Omega x$, since $\mathscr{X}_\Omega$ is discrete) such that $r \leq \mathscr{Y}(f_\Omega x', f_\Omega x)$ holds for every object (x', x) in $\mathscr{X}_r$ and every r.

This means precisely that $\mathscr{X}(x', x) \leq \mathscr{Y}(f_\Omega x', f_\Omega x)$ holds.

b. Given τ and τ', then

$$[\mathbb{N}^{op}, \mathscr{V}\text{-}\mathsf{cat}](N, \mathscr{V}\text{-}\mathsf{cat}(D^{\mathscr{V}} D_{\mathscr{X}} -, \mathscr{Y}))(\tau, \tau') = \bigwedge_x \mathscr{Y}(f_\Omega x, f'_\Omega x)$$

where f_Ω corresponds to τ and f'_Ω corresponds to τ'.

From the above, it follows that the $\mathscr{V}$-functor $q_{\mathscr{X}} : \mathscr{X}_\Omega \longrightarrow \mathscr{X}$ that sends each object x to itself is the couniversal such "cocone". More precisely, $r \leq \mathscr{X}(q_{\mathscr{X}} x', q_{\mathscr{X}} x)$ holds for every (x', x) in $\mathscr{X}_r$ and every r.

Furthermore, given any $\mathscr{V}$-functor $f_\Omega : \mathscr{X}_\Omega \longrightarrow \mathscr{Y}$ with the above properties, then there is a unique $\mathscr{V}$-functor $f_\Omega^\sharp : \mathscr{X} \longrightarrow \mathscr{Y}$ such that $f_\Omega^\sharp q_{\mathscr{X}} = f_\Omega$ holds.

The "2-dimensional aspect" of the colimit says that

$$\bigwedge_x \mathscr{Y}(f_\Omega^\sharp x, f_\Omega'^\sharp x) = \bigwedge_x \mathscr{Y}(f_\Omega x, f'_\Omega x)$$

Hence we have proved that $\mathscr{X}$ is isomorphic to $N * (D^{\mathscr{V}} D_{\mathscr{X}})$.

2. Suppose $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is given.

a. We first define a $\mathscr{V}\text{-}\mathsf{cat}$-functor $H^\sharp : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$.

To make the notation less heavy, for every small $\mathscr{V}$-category $\mathscr{X}$ and every $r \in \mathscr{V}_o$, we denote by X_r the *set* of pairs (x', x) such that $r \leq \mathscr{X}(x', x)$ and by X_Ω the *set* of objects of $\mathscr{X}$. Analogously, for a $\mathscr{V}$-functor $f : \mathscr{X} \longrightarrow \mathscr{Y}$, we denote by $f_r : X_r \longrightarrow Y_r$ and $f_\Omega : X_\Omega \longrightarrow Y_\Omega$ the maps corresponding to $(x', x) \mapsto (fx', fx)$ and the object assignment of f, respectively. Let also denote $d_0^r = D_{\mathscr{X}} \delta_0^r$ and $d_1^r = D_{\mathscr{X}} \delta_1^r$.

For every small $\mathscr{V}$-category $\mathscr{X}$, we put $H^\sharp \mathscr{X}$ to be the colimit $N * (H D_{\mathscr{X}})$.

Unravelling the definition of the weighted colimit, the 1-dimensional aspect says that to give a $\mathscr{V}$-functor $f^\sharp : H^\sharp \mathscr{X} \longrightarrow \mathscr{Y}$ is the same as to give a $\mathscr{V}$-functor $f : HX_\Omega \longrightarrow \mathscr{Y}$ such that

$$r \leq \bigwedge_{C \in HX_r} \mathscr{Y}(fHd_0^r(C), fHd_1^r(C)) \tag{2}$$

holds for all r.[3] In particular, there is a "quotient" $\mathscr{V}$-functor $c_{\mathscr{X}} : HX_\Omega \longrightarrow H^\sharp \mathscr{X}$

[3] By slight abuse of language, we shall use here and subsequently notation like $C \in HX_r$ to mean that C runs through all objects in *the $\mathscr{V}$-category* HX_r.

such that

$$r \leq \bigwedge_{C \in HX_r} H^\sharp \mathscr{X}(c_{\mathscr{X}} H d_0^r(C), c_{\mathscr{X}} H d_1^r(C)) \tag{3}$$

holds for all r, with the property that any $\mathscr{V}$-functor $HX_\Omega \longrightarrow \mathscr{Y}$ satisfying (2) uniquely factorizes through $c_{\mathscr{X}}$.

The 2-dimensional aspect of the colimit says that given any $f, g : HX_\Omega \longrightarrow \mathscr{Y}$, the relation

$$\bigwedge_{B \in HX_\Omega} \mathscr{Y}(f(B), g(B)) = \bigwedge_{A \in H^\sharp \mathscr{X}} \mathscr{Y}(f^\sharp(A), g^\sharp(A)) \tag{4}$$

holds.

For a $\mathscr{V}$-functor $f : \mathscr{X} \longrightarrow \mathscr{Y}$ we recall that the diagram

$$\begin{array}{ccc} X_r & \overset{d_1^r}{\underset{d_0^r}{\rightrightarrows}} & X_\Omega \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_\Omega} \\ Y_r & \overset{d_1^r}{\underset{d_0^r}{\rightrightarrows}} & Y_\Omega \end{array}$$

commutes serially. Hence f induces a $\mathscr{V}$-cat-natural transformation $D_f : D_{\mathscr{X}} \longrightarrow D_{\mathscr{Y}}$. Therefore we can define $H^\sharp f : H^\sharp \mathscr{X} \longrightarrow H^\sharp \mathscr{Y}$ as the unique mediating $\mathscr{V}$-functor

$$N * (HD_f) : N * (HD_{\mathscr{X}}) \longrightarrow N * (HD_{\mathscr{Y}})$$

In particular, we have the commutative diagram below:

$$\begin{array}{ccc} HX_\Omega & \xrightarrow{c_{\mathscr{X}}} & H^\sharp \mathscr{X} \\ {\scriptstyle Hf_\Omega}\downarrow & & \downarrow{\scriptstyle H^\sharp f} \\ HY_\Omega & \xrightarrow{c_{\mathscr{Y}}} & H^\sharp \mathscr{Y} \end{array}$$

Also, from the 2-dimensional aspect of the colimit (see Eq. (4)), we have that for any $f, g : \mathscr{X} \longrightarrow \mathscr{Y}$, the equality below holds:

$$\bigwedge_{B \in HX_\Omega} H^\sharp \mathscr{Y}(c_{\mathscr{Y}} Hf_\Omega(B), c_{\mathscr{Y}} Hg_\Omega(B)) = \bigwedge_{A \in H^\sharp \mathscr{X}} H^\sharp \mathscr{Y}(H^\sharp f(A), H^\sharp g(A)) \tag{5}$$

It remains to prove that the inequality

$$\mathscr{V}\text{-cat}(\mathscr{X}, \mathscr{Y})(f, g) \leq \mathscr{V}\text{-cat}(H^\sharp \mathscr{X}, H^\sharp \mathscr{Y})(H^\sharp f, H^\sharp g)$$

is satisfied. To that end, suppose that $r \leq \mathscr{V}\text{-cat}(\mathscr{X}, \mathscr{Y})(f, g)$ holds. This is equivalent to the fact that there is a mapping $t : X_\Omega \longrightarrow Y_r$ such that the triangles

$$\begin{array}{ccc} X_\Omega & \xrightarrow{t} & Y_r \\ & {\scriptstyle f_\Omega}\searrow & \downarrow{\scriptstyle d_0^r} \\ & & Y_\Omega \end{array} \qquad\qquad \begin{array}{ccc} X_\Omega & \xrightarrow{t} & Y_r \\ & {\scriptstyle g_\Omega}\searrow & \downarrow{\scriptstyle d_1^r} \\ & & Y_\Omega \end{array} \tag{6}$$

commute. In fact, $t(x) = (f(x), g(x))$. To prove that $r \leq \mathscr{V}\text{-cat}(H^\sharp \mathscr{X}, H^\sharp \mathscr{Y})(H^\sharp f, H^\sharp g)$ holds, we need to prove the inequality

$$r \leq \bigwedge_{A \in H^\sharp \mathscr{X}} H^\sharp \mathscr{Y}(H^\sharp f(A), H^\sharp g(A))$$

This follows from:

$$\begin{aligned} r &\leq \bigwedge_{C \in HY_r} H^\sharp \mathscr{Y}(c_{\mathscr{Y}} H d_0^r(C), c_{\mathscr{Y}} H d_1^r(C)) && \text{by (3)} \\ &\leq \bigwedge_{B \in HX_\Omega} H^\sharp \mathscr{Y}(c_{\mathscr{Y}} H d_0^r Ht(B), c_{\mathscr{Y}} H d_1^r Ht(B)) \end{aligned}$$

$$= \bigwedge_{B \in HX_\Omega} H^\sharp \mathscr{Y}(c_{\mathscr{Y}} H f_\Omega(B), c_{\mathscr{Y}} H g_\Omega(B)) \qquad \text{by (6)}$$

$$= \bigwedge_{A \in H^\sharp \mathscr{X}} H^\sharp \mathscr{Y}(H^\sharp f(A), H^\sharp g(A)) \qquad \text{by (5)}$$

We proved that $\mathscr{X} \mapsto H^\sharp \mathscr{X}$ can be extended to a $\mathscr{V}$-cat-functor $H^\sharp : \mathscr{V}\text{-cat} \longrightarrow \mathscr{V}\text{-cat}$.

b. We prove now that $H^\sharp \cong \mathrm{Lan}_{D^\mathscr{V}} H$ holds.

Due to the definition of $H^\sharp$, there is a $\mathscr{V}$-cat-natural isomorphism $\alpha : H \longrightarrow H^\sharp D^\mathscr{V}$. We prove that α is the unit of a left Kan extension.

Suppose that $K : \mathscr{V}\text{-cat} \longrightarrow \mathscr{V}\text{-cat}$ is any $\mathscr{V}$-cat-functor. To give a $\mathscr{V}$-cat-natural transformation $\tau : H^\sharp \longrightarrow K$ is to give a collection $\tau_\mathscr{X} : H^\sharp \mathscr{X} \longrightarrow K\mathscr{X}$ of $\mathscr{V}$-functors such that the square

$$\begin{array}{ccc} H^\sharp \mathscr{X} & \xrightarrow{\tau_\mathscr{X}} & K\mathscr{X} \\ {\scriptstyle H^\sharp f}\downarrow & & \downarrow{\scriptstyle Kf} \\ H^\sharp \mathscr{Y} & \xrightarrow[\tau_\mathscr{Y}]{} & K\mathscr{Y} \end{array}$$

commutes for every $\mathscr{V}$-functor $f : \mathscr{X} \longrightarrow \mathscr{Y}$.

The composite

$$H \xrightarrow{\alpha} H^\sharp D^\mathscr{V} \xrightarrow{\tau D^\mathscr{V}} K D^\mathscr{V}$$

yields a natural transformation $\tau^\flat : H^\sharp \longrightarrow K D^\mathscr{V}$.

Conversely, for every natural transformation $\sigma : H \longrightarrow K D^\mathscr{V}$, we define $\sigma^\sharp : H^\sharp \longrightarrow K$ at a $\mathscr{V}$-category $\mathscr{X}$ by considering first the composite

$$H D_\mathscr{X} \xrightarrow{\sigma D_\mathscr{X}} K D^\mathscr{V} D_\mathscr{X} \xrightarrow{K c_\mathscr{X}} K\mathscr{X}$$

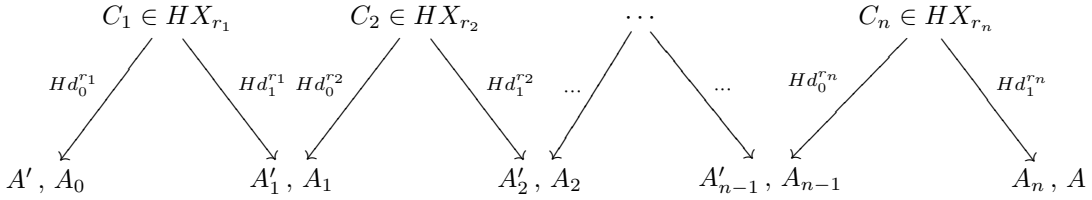
which yields $\sigma^\sharp_\mathscr{X} : H^\sharp \mathscr{X} \longrightarrow K\mathscr{X}$ by the passage to colimit (where $c_\mathscr{X} : D^\mathscr{V} D_\mathscr{X} \longrightarrow \mathscr{X}$ is the colimiting cocone).

The processes $\tau \mapsto \tau^\flat$ and $\sigma \mapsto \sigma^\sharp$ are inverses to each other. ◀

▶ **Remark 3.4.** The proof of the above theorem also provides a recipe on how to compute the left Kan extension of a $\mathscr{V}$-cat-functor $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-cat}$ along $D^\mathscr{V}$. Recall the notation such as X_Ω and X_r from item 2.a of the proof. For a $\mathscr{V}$-category $\mathscr{X}$, $H^\sharp \mathscr{X}$ is the $\mathscr{V}$-category having the same objects as HX_Ω (that is, the underlying set of objects of the $\mathscr{V}$-category obtained by applying H to the set of objects of $\mathscr{X}$). The couniversal cocone $c_\mathscr{X} : HX_\Omega \longrightarrow H^\sharp \mathscr{X}$ is the identity on objects. The $\mathscr{V}$-homs are, for any two objects A', A, given by $H^\sharp \mathscr{X}(A', A) =$

$$\bigvee \{HX_\Omega(A', A_0) \otimes r_1 \otimes HX_\Omega(A'_1, A_1) \otimes r_2 \otimes \ldots \otimes HX_\Omega(A'_{n-1}, A_{n-1}) \otimes r_n \otimes HX_\Omega(A_n, A)\}$$

where the join is computed over all (possibly empty) paths $(A_0, A'_1, A_1, \ldots, A'_n, A_n)$ and all (possibly empty) tuples of elements $(r_1, \ldots, r_n)$ such that there are $C_i \in HX_{r_i}$ with $Hd_0^{r_i}(C_i) = A_{i-1}$, $Hd_1^{r_i}(C_i) = A'_i$, for all $i = 1, n$:



▶ **Corollary 3.5.** *Every* $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ *has a* $\mathscr{V}$-cat-*ification.*

Proof. Apply Theorem 3.3 to the composite $H = D^{\mathscr{V}}T : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$. ◀

In particular, we obtain from the above that $\mathsf{Id} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is the $\mathscr{V}$-cat-ification of $\mathsf{Id} : \mathsf{Set} \longrightarrow \mathsf{Set}$. Thus by [12, Theorem 5.1],

▶ **Proposition 3.6.** The $\mathscr{V}$-cat-functor $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is dense.

Corollary 3.5, together with the proof of Theorem 3.3 (see the above remark), give us a recipe of how to compute various $\mathscr{V}$-cat-ifications.

▶ **Examples 3.7 (The $\mathscr{V}$-cat-ification of polynomial functors).**

1. Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, $TX = S$ be a constant functor. Then $T_{\mathscr{V}}$ is again constant, where $T_{\mathscr{V}}\mathscr{X} = D^{\mathscr{V}}S$ for any $\mathscr{V}$-category $\mathscr{X}$.
2. Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be the functor $TX = X^n$, for n a natural number. Then $T_{\mathscr{V}}$ maps a $\mathscr{V}$-category $\mathscr{X}$ to its n-th power $\mathscr{X}^n$, where an easy computation shows
$$\mathscr{X}^n((x'_0,\dots,x'_{n-1}),(x_0,\dots,x_{n-1})) = \mathscr{X}(x'_0,x_0) \wedge \cdots \wedge \mathscr{X}(x'_{n-1},x_{n-1}).$$
3. If n is an *arbitrary* cardinal number, the $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ of $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, $TX = X^n$ also exists and $T_{\mathscr{V}}\mathscr{X}((x'_i),(x_i)) = \bigwedge_i \mathscr{X}(x'_i,x_i)$. That is, $T_{\mathscr{V}}\mathscr{X} = \mathscr{X}^n$.
4. The $\mathscr{V}$-cat-ification of a finitary polynomial functor $X \mapsto \coprod_n X^n \bullet \Sigma n$ is the "strongly polynomial" $\mathscr{V}$-cat-functor $\mathscr{X} \mapsto \coprod_n \mathscr{X}^n \otimes D^{\mathscr{V}}\Sigma n$, where n ranges through finite sets.

▶ **Example 3.8 (The $\mathscr{V}$-cat-ification of the powerset).** Let $P : \mathsf{Set} \longrightarrow \mathsf{Set}$ be the powerset functor. By Theorem 3.3 and Corollary 3.5, its $\mathscr{V}$-cat-ification $P_{\mathscr{V}}$ is defined as follows. Let $\mathscr{X}$ be any small $\mathscr{V}$-category. Then the objects of $P_{\mathscr{V}}\mathscr{X}$ are subsets of the set of objects of $\mathscr{X}$, while the $\mathscr{V}$-"distances" in $P_{\mathscr{V}}\mathscr{X}$ are computed as follows:

$$\begin{aligned} P_{\mathscr{V}}\mathscr{X}(A',A) &= \bigvee_s \{s \mid \text{ there is } B \text{ in } PX_s \text{ s.t. } Pd_0^s(B) = A' \text{ and } Pd_1^s(B) = A\} \\ &= \bigvee_s \{s \mid \forall\, x' \in A'\ \exists\, x \in A.\, s \le \mathscr{X}(x',x) \text{ and } \forall\, x \in A\ \exists\, x' \in A'.\, s \le \mathscr{X}(x',x)\ \} \end{aligned}$$

If the quantale $\mathscr{V}$ is constructively completely distributive [7, 19], as it is the case with $\mathscr{V} = [0,1]$ and $\mathscr{V} = [0,\infty]$, then the above is equivalent to the following:

$$\sup\{\ \sup_{x' \in A'}\ \inf_{x \in A}\ \mathscr{X}(x',x)\ ,\ \sup_{x \in A}\ \inf_{x' \in A'}\ \mathscr{X}(x',x)\} \tag{7}$$

where we switched notation to the dual order (that is, the natural "less-or-equal" order in case of reals). So we write inf for $\bigvee$ and sup for $\bigwedge$, in order to emphasise the interpretation of $\mathscr{V}$-cat as metric spaces.

Recall that this metric is known as the Pompeiu-Hausdorff metric ([9, §28], [15, §21]).

We should mention also the connection with the work of [1]. Finally, observe that in case $\mathscr{V} = 2$ (ie $\mathscr{V}$-cat = Preord), the above specializes to the locally monotone functor $P_2 : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ which sends a preorder $(X,\le)$ to the Egli-Milner preorder

$$A' \sqsubseteq A \quad \text{iff} \quad \forall\, x' \in A'\ \exists\, x \in A.\, x' \le x \text{ and } \forall\, x' \in A\ \exists\, x \in A'.\, x' \le x$$

on the powerset PX.

▶ **Remark 3.9.** The $\mathscr{V}$-cat-functor $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ preserves conical colimits. This follows from the $D_o^{\mathscr{V}}$ being an *ordinary* left adjoint. However, the $\mathscr{V}$-cat-functor $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ is not a left $\mathscr{V}$-cat-*adjoint*, as its ordinary right adjoint functor $V^{\mathscr{V}}$ cannot be extended to a $\mathscr{V}$-cat-functor.

▶ **Proposition 3.10.** *The assignment* $(-)_{\mathscr{V}} : [\mathsf{Set},\mathsf{Set}] \longrightarrow [\mathscr{V}\text{-}\mathsf{cat},\mathscr{V}\text{-}\mathsf{cat}]$, $T \mapsto T_{\mathscr{V}}$ *of the* $\mathscr{V}$-cat-*ification preserves all colimits preserved by* $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$. *In particular,* $T \mapsto T_{\mathscr{V}}$ *preserves conical colimits.*

Proof. Any natural transformation $\tau : T \longrightarrow S$ induces a $\mathscr{V}$-cat-natural transformation

$$(\tau_{\mathscr{V}})_{\mathscr{X}} = N * (D^{\mathscr{V}} \tau D_{\mathscr{X}}) : N * (D^{\mathscr{V}} T D_{\mathscr{X}}) \longrightarrow N * (D^{\mathscr{V}} S D_{\mathscr{X}})$$

Since any colimit is cocontinuous in its weight and since

$$N * (D^{\mathscr{V}} T D_{\mathscr{X}}) \cong (D^{\mathscr{V}} T D_{\mathscr{X}}) * N$$

holds, the assignment $T \mapsto T_{\mathscr{V}}$ preserves all colimits that are preserved by $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}$-$\mathsf{cat}$. The last statement follows from Remark 3.9. ◀

▶ **Corollary 3.11.** *Suppose that the coequalizer*

$$T_\Gamma \underset{\rho}{\overset{\lambda}{\rightrightarrows}} T_\Sigma \xrightarrow{\gamma} T$$

is the equational presentation of a λ-accessible functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$. Then the $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ can be obtained as the coequalizer

$$(T_\Gamma)_{\mathscr{V}} \underset{\rho_{\mathscr{V}}}{\overset{\lambda_{\mathscr{V}}}{\rightrightarrows}} (T_\Sigma)_{\mathscr{V}} \xrightarrow{\gamma_{\mathscr{V}}} T_{\mathscr{V}}$$

in $[\mathscr{V}\text{-}\mathsf{cat}, \mathscr{V}\text{-}\mathsf{cat}]$.

Proof. A coequalizer is a conical colimit. Now use Proposition 3.10. ◀

▶ **Remark 3.12 (The $\mathscr{V}$-cat-ification of finitary functors).** Corollary 3.11 allows us to say that the $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ of a finitary functor T is given by imposing the "same" operations and equations in $\mathscr{V}$-cat.

Intuitively, the endofunctors on $\mathscr{V}$-cat that arise as left Kan extensions along the discrete functor D^V are the $\mathscr{V}$-cat-endofunctors definable in "discrete arities". This statement will be made formal in future work, here we restrict ourselves to a basic example.

▶ **Example 3.13.** Consider a set A and the associate stream functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, $TX = X \times A$. If A carries the additional structure of a $\mathscr{V}$-category (that, is, there is a $\mathscr{V}$-category $\mathscr{A}$ with underlying set of objects A), then T_o can be written as the composite $V^{\mathscr{V}} H$, where $H : \mathsf{Set} \longrightarrow \mathscr{V}$-$\mathsf{cat}$ is the $\mathscr{V}$-cat-functor $HX = D^{\mathscr{V}} X \otimes \mathscr{A}$. Now it is immediate to see that the latter extends to the stream functor $H^\sharp$ on $\mathscr{V}$-cat over the "generalized metric space" $\mathscr{A}$, mapping a $\mathscr{V}$-category $\mathscr{X}$ to the tensor product of $\mathscr{V}$-categories $H^\sharp \mathscr{X} = \mathscr{X} \otimes \mathscr{A}$.

The above example is typical. It happens quite often for endofunctors on Set to carry an interesting $\mathscr{V}$-metric where TX is a $\mathscr{V}$-category rather than a mere set, for every X, and this structure is compatible with substitution. The following generalizes the notion of an order on a functor [11] from $\mathscr{V} = \mathbb{2}$.

▶ **Definition 3.14.** Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a functor. We say that T *carries a $\mathscr{V}$-metric* if there is a $\mathscr{V}$-cat-functor $H : \mathsf{Set} \longrightarrow \mathscr{V}$-$\mathsf{cat}$ such that T coincides with the composite

$$\mathsf{Set}_o \xrightarrow{H_o} \mathscr{V}\text{-}\mathsf{cat}_o \xrightarrow{V^{\mathscr{V}}} \mathsf{Set}_o \ .$$

Let T and H be as in the above definition. How are T and $H^\sharp$, the left Kan extension of H along $D^{\mathscr{V}}$ as provided by Theorem 3.3, related? As $D^{\mathscr{V}}$ is fully faithful, the unit $H \longrightarrow H^\sharp D^{\mathscr{V}}$ of the left Kan extension is a $\mathscr{V}$-cat-natural isomorphism. Hence $T_o = V^{\mathscr{V}} H_o \cong V^{\mathscr{V}} H^\sharp D^{\mathscr{V}}$; using now the counit of the ordinary adjunction $D_o^{\mathscr{V}} \dashv V^{\mathscr{V}}$, we obtain an ordinary natural transformation

$$\beta : T_o V^{\mathscr{V}} \longrightarrow V^{\mathscr{V}} H_o^\sharp : \mathscr{V}\text{-}\mathsf{cat}_o \longrightarrow \mathsf{Set}_o.$$

▶ **Proposition 3.15.** *The natural transformation β is component-wise bijective.*

Consequently, $H^\sharp$ is a lifting of T to $\mathscr{V}$-cat.

▶ **Example 3.16 (The Kantorovich lifting).** Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a functor and let $\heartsuit : T\mathscr{V} \longrightarrow \mathscr{V}$ be a map (a $\mathscr{V}$-valued predicate lifting), where by slight abuse we identify the quantale with its underlying set of elements. We ask for $\heartsuit$ to be $\mathscr{V}$-monotone, in the following sense: for every set X and maps $h, k : X \longrightarrow \mathscr{V}$, the inequality

$$\bigwedge_{x\in X} [h(x), k(x)] \leq \bigwedge_{A\in TX} [\heartsuit(T(h)(A)), \heartsuit(T(k)(A))]$$

should hold.[4] Using the $\mathscr{V}$-valued predicate lifting $\heartsuit$, we can endow T with a $\mathscr{V}$-metric as follows: for each set X, put HX to be the $\mathscr{V}$-category with set of objects TX, and $\mathscr{V}$-distances

$$(HX)(A', A) = \bigwedge_{h:X\longrightarrow\mathscr{V}} [\heartsuit(T(h)(A')), \heartsuit(T(h)(A))]$$

where A', A are elements of TX. For a function $f : X \longrightarrow Y$, we let Hf act as Tf on objects. It is easy to see that the above defines indeed a $\mathscr{V}$-metric for T, that is, a $\mathscr{V}$-cat-functor $H : \mathsf{Set} \longrightarrow \mathscr{V}$-$\mathsf{cat}$ (the $\mathscr{V}$-cat-enrichment being a consequence of Set being free as a $\mathscr{V}$-cat-category) with $V^{\mathscr{V}}H_o = T$. The corresponding lifting $H^\sharp$ specializes to the Kantorovich lifting as defined in [4] in case $\mathscr{V} = [0, \infty]$. Explicitly, a $\mathscr{V}$-category $\mathscr{X}$ gets mapped to the small $\mathscr{V}$-category $H^\sharp\mathscr{X}$ with set of objects TX_Ω and $\mathscr{V}$-homs

$$H^\sharp\mathscr{X}(A', A) = \bigwedge_{h:\mathscr{X}\longrightarrow\mathscr{V}} [\heartsuit(T(h_\Omega)(A')), \heartsuit(T(h_\Omega)(A))]$$

for every A', A in TX_Ω, where this time h ranges over $\mathscr{V}$-functors.

4 Relating behaviours across different base categories

In the previous section, we have shown that every $\mathscr{V}$-cat-functor $H : \mathsf{Set} \longrightarrow \mathscr{V}$-$\mathsf{cat}$ has a left Kan extension along $D^{\mathscr{V}}$, denoted $H^\sharp$. Now, each such functor induces a set-endofunctor simply by forgetting the $\mathscr{V}$-cat-structure

$$\mathsf{Set}_o \xrightarrow{H_o} \mathscr{V}\text{-}\mathsf{cat} \xrightarrow{V^{\mathscr{V}}} \mathsf{Set}_o$$

In the special case when H is $D^{\mathscr{V}}T$, the above composite gives back T, and $H^\sharp$ is $T_{\mathscr{V}}$, the $\mathscr{V}$-cat-ification of T.

We plan to see how the corresponding behaviors are related. In particular, we show that if $T_{\mathscr{V}}$ is the $\mathscr{V}$-cat-ification of $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, then $T_{\mathscr{V}}$-behaviour and T-behaviour coincide under some conditions imposed on the base quantale $\mathscr{V}$. This requires comparing behaviours across different base categories.

▶ **Remark 4.1.** For each quantale $\mathscr{V}$, the inclusion (quantale morphism) $\mathsf{d} : 2 \longrightarrow \mathscr{V}$ given by $0 \mapsto 0, 1 \mapsto e$ has a right adjoint (as it preserves suprema), denoted $\mathsf{v} : \mathscr{V} \longrightarrow 2$ which maps an element r of $\mathscr{V}$ to 1 if $e \leq r$, and to 0 otherwise.[5]

This induces as usual the *change-of-base* adjunction (even a 2-adjunction, see [5])

$$2 \underset{\mathsf{v}}{\overset{\mathsf{d}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{V} \qquad \mapsto \qquad \mathsf{Preord} \underset{\mathsf{v}_*}{\overset{\mathsf{d}_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{V}\text{-}\mathsf{cat}$$

Explicitly, the functor d_* maps a preordered set X to the $\mathscr{V}$-category d_*X with same set of objects, and $\mathscr{V}$-homs given by $\mathsf{d}_*X(x', x) = e$ if $x' \leq x$, and $\bot$ otherwise. Its right adjoint transforms a $\mathscr{V}$-category $\mathscr{X}$ into the preorder $\mathsf{v}_*\mathscr{X}$ with same objects again, and order

[4] This generalizes the notion of a monotone predicate lifting from the two-elements quantale to arbitrary $\mathscr{V}$, see [3, Section 7].

[5] Notice that v is only a *lax* morphism of quantales, being right adjoint.

$x' \leq x$ iff $e \leq \mathscr{X}(x', x)$ holds. Hence d_*X is the free $\mathscr{V}$-category on the preorder X, while $v_*\mathscr{X}$ is the underlying ordinary category (which happens to be a preorder, due to simple nature of quantales) of the $\mathscr{V}$-category $\mathscr{X}$.

Note that d_* is both a $\mathscr{V}$-cat-functor and a Preord-functor, while its right adjoint v_* (in fact, the whole adjunction $\mathsf{d}_* \dashv \mathsf{v}_*$) is only Preord-enriched.

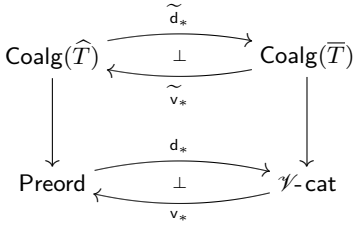
In case $\mathscr{V}$ is nontrivial, and e and $\top$ coincide (the quantale is *integral*), the embedding $\mathsf{d} : 2 \longrightarrow \mathscr{V}$ has also a left adjoint $\mathsf{c} : \mathscr{V} \longrightarrow 2$, given by $\mathsf{c}(r) = 0$ iff $r = \bot$, otherwise $\mathsf{c}(r) = 1$. Notice that c is only a colax morphism of quantales, in the sense that $\mathsf{c}(e) \leq 1$ (in fact, here we have equality!) and $\mathsf{c}(r \otimes s) \leq \mathsf{c}(r) \wedge \mathsf{c}(s)$, for all r, s in $\mathscr{V}$.

We shall in the sequel assume that c is actually a morphism of quantales. The reader can check that this boils down to the requirement that $r \otimes s = \bot$ in $\mathscr{V}$ implies $r = \bot$ or $s = \bot$. That is, the quantale has no *zero divisors*. All our examples satisfy this assumption.

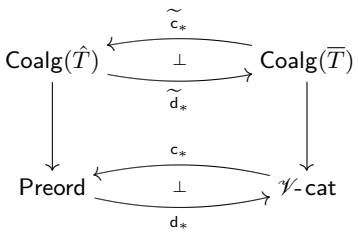
If this is the case, d_* also has a left adjoint c_* mapping a $\mathscr{V}$-category $\mathscr{X}$ to the preorder $\mathsf{c}_*\mathscr{X}$ with same objects, such that $x' \leq x$ iff $\mathscr{X}(x', x) \neq \bot$, and the adjunction $\mathsf{c}_* \dashv d_*$ is $\mathscr{V}$-cat-enriched:

From the above remark we obtain the following:

▶ **Proposition 4.2.** *Let $\mathscr{V}$ be an arbitrary quantale and let $\widehat{T} : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ be a locally monotone functor (that is, Preord-enriched) and $\overline{T} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ be a lifting of $\widehat{T}$ to $\mathscr{V}\text{-}\mathsf{cat}$ (meaning that $\overline{T}$ is $\mathscr{V}$-cat-functor such that $\mathsf{v}_*\overline{T} \cong \widehat{T}\mathsf{v}_*$ holds). Then the locally monotone adjunction $\mathsf{d}_* \dashv \mathsf{v}_*$ lifts to a locally monotone adjunction $\widetilde{\mathsf{d}}_* \dashv \widetilde{\mathsf{v}}_*$ between the associated Preord-categories of coalgebras.*



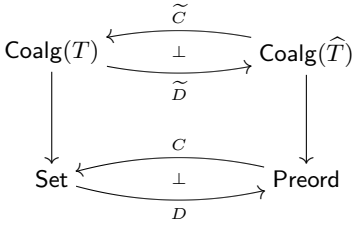
▶ **Proposition 4.3.** *Assume now that $\mathscr{V}$ is a non-trivial integral quantale without zero divisors. Let again $\widehat{T} : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ be a locally monotone functor, but this time consider $\overline{T} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ be an extension of $\widehat{T}$ to $\mathscr{V}\text{-}\mathsf{cat}$ (meaning that $\overline{T}$ is a $\mathscr{V}$-cat-functor, such that $\overline{T}\mathsf{d}_* \cong \widehat{T}\mathsf{d}_*$ holds). Then the $\mathscr{V}$-cat-adjunction $\mathsf{c}_* \dashv \mathsf{d}_*$ lifts to a $\mathscr{V}$-cat-adjunction $\widetilde{\mathsf{c}}_* \dashv \widetilde{\mathsf{d}}_*$ between the associated $\mathscr{V}$-cat-categories of coalgebras.*



We come back now to the discrete functor $D^{\mathscr{V}} : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$. It is easy to see that it decomposes as $\mathsf{d}_*D : \mathsf{Set} \to \mathsf{Preord} \to \mathscr{V}\text{-}\mathsf{cat}$. Additionally, recall the following (see also Example 3.2.2):

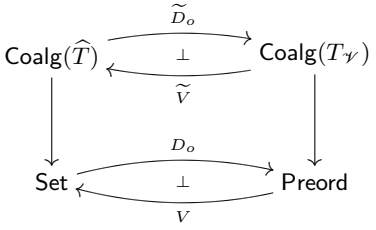
1. There are locally monotone functors $D : \mathsf{Set} \longrightarrow \mathsf{Preord}$, $C : \mathsf{Preord} \longrightarrow \mathsf{Set}$, where D maps a set to its discrete preorder and C maps a preorder to its set of connected components.
2. There is a chain $C_o \dashv D_o \dashv V : \mathsf{Preord} \longrightarrow \mathsf{Set}$ of ordinary adjunctions where V is the underlying-set forgetful functor.
3. The locally monotone adjunction $C \dashv D$ is $\mathscr{V}$-cat-enriched.

▶ **Lemma 4.4** ([2]). *Let* $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ *and* $\widehat{T} : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ *an extension of* T *(a locally monotone functor such that* $DT \cong \widehat{T}D$*). Then the locally monotone adjunction* $C \dashv D$ *lifts to a locally monotone adjunction* $\widetilde{C} \dashv \widetilde{D}$ *between the associated categories of coalgebras:*



Consequently, $\widetilde{D}$ will preserve limits, in particular, the final coalgebra (if it exists).

▶ **Lemma 4.5** ([2]). *Let* $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ *and* $\widehat{T} : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ *a lifting of* T *(an ordinary functor such that* $TV \cong V\widehat{T}$*). Then the ordinary adjunction* $D_o \dashv V$ *lifts to an ordinary adjunction* $\widetilde{D}_o \dashv \widetilde{V}$ *between the associated categories of coalgebras.*



Consequently, $\widetilde{V}$ will preserve limits; in particular, the underlying set of the final $\widehat{T}$-coalgebra (if it exists) will be the final T-coalgebra.

▶ **Remark 4.6.** We have shown in the previous section that $D^{\mathscr{V}} = \mathsf{d}_*D$ is $\mathscr{V}$-cat-dense. Using that D is fully faithful, it follows from [12, Theorem 5.13] that also d_* is $\mathscr{V}$-cat-dense and that $\mathsf{d}_* = \mathsf{Lan}_D(D^{\mathscr{V}})$ holds.

Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and denote by T_2 is 2-cat-ification, that is, its Preord-ification [3]. Then the $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ of T can be computed in two stages, as follows:

$$\begin{aligned} T_{\mathscr{V}} &= \mathsf{Lan}_{D^{\mathscr{V}}}(D^{\mathscr{V}}T) \\ &= \mathsf{Lan}_{(\mathsf{d}_*D)}(\mathsf{d}_*DT) = \mathsf{Lan}_{\mathsf{d}_*}(\mathsf{Lan}_D(\mathsf{d}_*DT)) \text{ by} \quad \text{[12, Theorem 4.47]} \\ &\cong \mathsf{Lan}_{\mathsf{d}_*}(\mathsf{Lan}_D(\mathsf{d}_*T_2D)) \text{ (because } DT \cong T_2D) \\ &\cong \mathsf{Lan}_{\mathsf{d}_*}(\mathsf{d}_*T_2) \text{ by} \quad \text{[12, Theorem 5.29]} \end{aligned}$$

where the last isomorphism holds because the composite d_*T_2 preserves all colimits $\mathsf{Preord}(D-, X) * D$, for X in Preord. To see this, notice first that T_2 does so by construction, while for d_* it follows from being $\mathsf{Lan}_D(D^{\mathscr{V}}) = \mathsf{Lan}_D(\mathsf{d}_*D)$, again using [12, Theorem 5.29].

The above simply says that

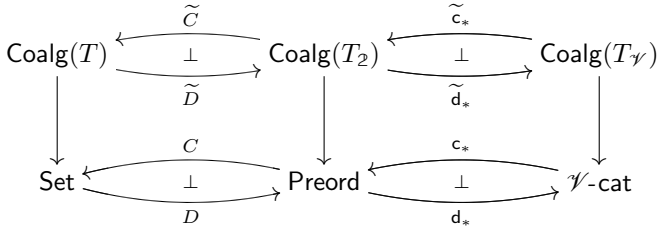
The $\mathscr{V}$-cat*-ification of an endofunctor* T *of* Set *can be obtained as taking first the* Preord*-ification* $T_2 : \mathsf{Preord} \longrightarrow \mathsf{Preord}$, [6] *then computing the left Kan extension along* $\mathsf{d}_* : \mathsf{Preord} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ *of the composite* $\mathsf{Preord} \xrightarrow{T_2} \mathsf{Preord} \xrightarrow{\mathsf{d}_*} \mathscr{V}\text{-}\mathsf{cat}$.

Putting things together we now obtain

▶ **Theorem 4.7.** *Let* $\mathscr{V}$ *be a non-trivial integral quantale without zero divisors, and* $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ *an arbitrary endofunctor, with* $\mathscr{V}$-cat*-ification* $T_{\mathscr{V}} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$. *Then the* $\mathscr{V}$-cat*-adjunctions* $C \dashv D : \mathsf{Set} \longrightarrow \mathsf{Preord}$, $\mathsf{c}_* \dashv \mathsf{d}_* : \mathsf{Preord} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ *lift to*

[6] Which has been considered in [3]; note in particular that T_2 is also a lifting of T to Preord.

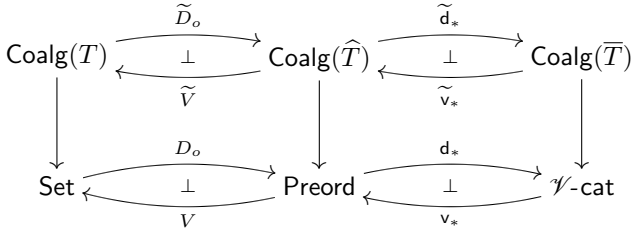
$\mathscr{V}$-cat-adjunctions between the associated $\mathscr{V}$-cat-categories of coalgebras:



Since the $\mathscr{V}$-cat-ification $T_{\mathscr{V}}$ of an endofunctor T on Set is supposed to be "T in the world of $\mathscr{V}$-categories", the theorem above confirms the expectation that final $T_{\mathscr{V}}$-coalgebras have a discrete metric. In fact, we can say that the final T-coalgebra is the final $T_{\mathscr{V}}$-coalgebra, if we consider Coalg(T) as a full (enriched-reflective) subcategory of Coalg$(T_{\mathscr{V}})$.

The next theorem deals with a more general situation where the final metric-coalgebra is the final set-coalgebra with an additional metric. This includes in particular the case where $\overline{T}$ is $H^\sharp$ for some $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ with $V^{\mathscr{V}} H_o = T_o$.

▶ **Theorem 4.8.** *Let $\mathscr{V}$ be a quantale, $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be an arbitrary endofunctor, $\widehat{T} : \mathsf{Preord} \longrightarrow \mathsf{Preord}$ a lifting of T to* Preord*, and $\overline{T} : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ be a lifting of $\widehat{T}$ to $\mathscr{V}$-cat. Then the ordinary adjunction $D_o \dashv V : \mathsf{Set} \longrightarrow \mathsf{Preord}$, respectively the* Preord*-adjunction $\mathsf{d}_* \dashv \mathsf{v}_* : \mathsf{Preord} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ lift to adjunctions between the associated $\mathscr{V}$-cat-categories of coalgebras:*



▶ **Example 4.9.** Recall from Example 3.13 the stream functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, $TX = X \times A$, and its lifting $H^\sharp : \mathscr{V}\text{-}\mathsf{cat} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$, $H^\sharp \mathscr{X} = \mathscr{X} \otimes \mathscr{A}$. Assume that the quantale is integral. Then the final coalgebra is the $\mathscr{V}$-category $\mathscr{A}^{\otimes\infty}$ having streams over A as objects, with $\mathscr{V}$-distances

$$\mathscr{A}^{\otimes\infty}((a_n)_n, (b_n)_n) = \bigwedge_n \{\mathscr{A}(a_0, b_0) \otimes \mathscr{A}(a_1, b_1) \otimes \ldots \otimes \mathscr{A}(a_n, b_n)\}$$

If $\mathscr{V}$ is the real half-line from Example 2.1.2, and $\mathscr{A}$ is the two-elements metric space $\{0, 1\}$ with $\mathscr{V}$-distances $\mathscr{A}(0, 1) = \mathscr{A}(1, 0) = 1$, $\mathscr{A}(0, 0) = \mathscr{A}(1, 1) = 0$, we obtain that the $\mathscr{V}$-distance between two streams is n iff they are different on at most n positions.

5 Conclusions

We showed that every functor $H : \mathsf{Set} \longrightarrow \mathscr{V}\text{-}\mathsf{cat}$ has a left-Kan extension $H^\sharp$, and that the final $H^\sharp$-coalgebra is the final $V^{\mathscr{V}} H_o$-coalgebra equipped with a $\mathscr{V}$-metric. In the case where H takes only discrete values, the final coalgebra is discrete as well.

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References

1 A. Akhvlediani, M. M. Clementino and W. Tholen, On the categorical meaning of the Hausdorff and Gromov distances I, *Topology and its Applic.* 157(8) (2010), 1275–1295

2 A. Balan and A. Kurz, Finitary functors: from Set to Preord and Poset. In: A. Corradini et al (eds.), *CALCO 2011*, LNCS 6859, Springer, Heidelberg (2011), 85–99

3 A. Balan, A. Kurz and J. Velebil, Positive fragments of coalgebraic logics, accepted for publication in *Logic. Meth. Comput. Sci.* (2015), available at `http://arxiv.org/pdf/1402.5922v1.pdf`

4 P. Baldan, F. Bonchi, H. Kerstan and B. König, Behavioral metrics via functor lifting. In: V. Raman and S. P. Suresh (eds.), *FSTTCS2014*, LIPIcs 29 (2014) 403–415

5 S. Eilenberg, and G. M. Kelly, Closed categories. In: S. Eilenberg et al (eds.), *Proceedings of the Conference on Categorical Algebra*, Springer, Berlin Heidelberg (1966), 421–562

6 B. Flagg and R. Kopperman, Continuity spaces: reconciling domains and metric spaces, *Theoret. Comput. Sci.* 177(1) (1997), 111–138

7 P. Freyd and A. Scedrov. *Categories, Allegories*, North Holland (1990)

8 H. Gaifman, V. Pratt, Partial order models of concurrency and the computation of functions. In: *Proceedings of the Symposium on Logic in Computer Science (LICS '87)*, Ithaca, NY (1987), 72–85

9 F. Hausdorff, *Mengenlehre*. 3rd edition, de Gruyter (1935)

10 D. Hofmann and C. D. Reis, Probabilistic metric spaces as enriched categories, *Fuzzy Sets and Systems* 210 (2013), 1–21

11 J. Hughes and B. Jacobs, Simulations in coalgebra, *Theor. Comput. Sci.* 327(1-2): 71-108 (2004)

12 G. M. Kelly, *Basic concepts of enriched category theory*, London Math. Soc. Lecture Notes Series 64, Cambridge Univ. Press (1982) also available as *Repr. Theory Appl. Categ.* 10 (2005)

13 G. M. Kelly and S. Lack, $\mathscr{V}$-cat is locally presentable or bounded if $\mathscr{V}$ is so, *Theory Appl. Categ.* 8(23) (2001), 555–575

14 F. W. Lawvere, Metric spaces, generalized logic, and closed categories, *Rendiconti del Seminario Matematico e Fisico di Milano* XLIII (1973), 135–166, also available as *Repr. Theory Appl. Categ.* 1 (2002), 1–37

15 D. Pompeiu, Sur la continuité des fonctions des variables complexes, *Ann. Fac. Sci. Toulouse* 2(7) (1905), 265–315, available at `http://www.numdam.org/item?id=AFST_1905_2_7_3_265_0`

16 J. J. M. M. Rutten, Elements of generalized ultrametric domain theory, *Theoret. Comput. Sci.* 170 (1996), 349–381

17 J. J. M. M. Rutten. Relators and metric bisimulations (extended abstract). In: B. Jacobs et al (eds.), *CMCS'98, Electr. Notes Theor. Comput. Sci.* 11 (1998), 1–7

18 J. J. M. M. Rutten, Universal coalgebra: a theory of systems, *Theoret. Comput. Sci.* 249 (2000), 3–80

19 R. Wood, Ordered sets via adjunctions. In: M.-C. Pedicchio and W. Tholen (eds.), *Categorical Foundations*. Cambridge Univ. Press (2004), 5–47

20 J. Worrell, Coinduction for recursive data types: partial order, metric spaces and Ω-categories. In: H. Reichel (ed.), *CMCS'2000, Electr. Notes Theor. Comput. Sci.* 33 (2000), 337–356

21 J. Worrell, On coalgebras and final semantics. PhD thesis, University of Oxford (2000), available at `http://www.cs.ox.ac.uk/people/james.worrell/thesis.ps`