Equations and Coequations for Weighted Automata*

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Weighted automata are a generalization of non-deterministic automata introduced by Schützenberger [10]. Every transition is associated with an input letter from an alphabet A and a weight expressing the cost (or probability, time, resources needed) of its execution. This weight is typically an element of a semiring. The multiplication of the semiring is used to accumulate the weight of a path by multiplying the weights of each transition in the path, while the addition of the semiring computes the weight of a string w by summing up the weights of the paths labeled with w [8]. In this way, the behaviour of weighted automata is given in terms of formal power series, i.e. functions assigning a weight to each finite string w over A.

Weighted automata may have a non-deterministic behaviour because different transitions from the same state may be labeled by the same input letter, with possibly different weights. However, they can be determinized by assigning a linear structure to the state-space using a generalization of the powerset construction for non-deterministic automata [4]. As such, determinized weighted automata are typically infinite-state, but determinization allows us to study weighted automata both from an algebraic perspective and a coalgebraic one. From the algebraic perspective, a (determinized) weighted automaton is just an algebra with a unary operation for each input symbol, whereas coalgebraically, a weighted automaton is a deterministic transition system with output weights associated to each state.

In this context, and building on the work by [2] on ordinary deterministic automata and on the duality between reachability and observability [1, 5, 3], we study equations and coequations for weighted automata. In general, equations characterize classes of algebras called varieties [6], whereas coequations characterize classes of coalgebras, so-called covarieties [9]. Using the algebraic perspective, an equation for a weighted automaton is just a pair of words (u, v). Satisfaction becomes reachability: a weighted automaton satisfies an equation (u, v) if from any state, the linear combination of states reached after reading u is the same as the one reached after reading v [2].

Dually, the coalgebraic perspective allows us to define coequations for weighted automata as subsets (or predicates) of power series. A weighted automaton M satisfies a coequation S if for any function associating an output weight to each state, the behaviour of M from any initial state is a power series in S.

Based on the definition of equations and coequations we showed how to get the maximun set of equations and the minimum set of coequations satisfied by a given weighted automata. This process is similar to the one described in [2] for deterministic automata in which a simplification can be made if we take into account the linear structure of a (determinized) weighted automata.

We established a duality result between sets of equations called congruences and sets of coequations called closed subsystems. As a consequence, we prove that classes of weighted automata defined by congruences are in a one-to-one correspondence with classes of weighted automata defined by closed subsystems. This allows, for example, to give equational characterizations to some specific subsets of power series and, vice versa, to define the least

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subset of power series specified by a set of equations.

Using the linear structure of a weighted automata we can define a more general kind of equations and congruences, called linear, and ask whether a duality result holds or not. In general, a full duality result does not hold anymore, but when the weights come from a field, we still have one direction of it: a variety defined by a linear congruence can be recovered from a corresponding covariety. As an example, we can show that linear congruences (under certain conditions) are finitely generated by a set of linear equations, using the fact that by Hilbert basis theorem [7, VIII, Theorem 4.9], linear congruences correspond to ideals of polynomials.

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