

From Farkas' lemma to linear programming

an exercise in diagrammatic algebra

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Diagrammatic Polyhedral Algebra Syntax

$$c ::= \bullet \mid \text{---}\bullet\text{---} \mid \text{---}\square[k]\text{---} \mid \text{---}\circ\text{---} \mid \circ\text{---} \mid \quad (1)$$

$$\bullet\text{---} \mid \text{---}\bullet\text{---} \mid \text{---}\square(k)\text{---} \mid \text{---}\circ\text{---} \mid \text{---}\circ\text{---} \mid \quad (2)$$

$$\text{---}\square[\geq]\text{---} \mid \quad (3)$$

$$\vdash \mid \quad (4)$$

$$\square \mid - \mid \times \mid c;c \mid c \oplus c \quad (5)$$

Diagrammatic Polyhedral Algebra Syntax

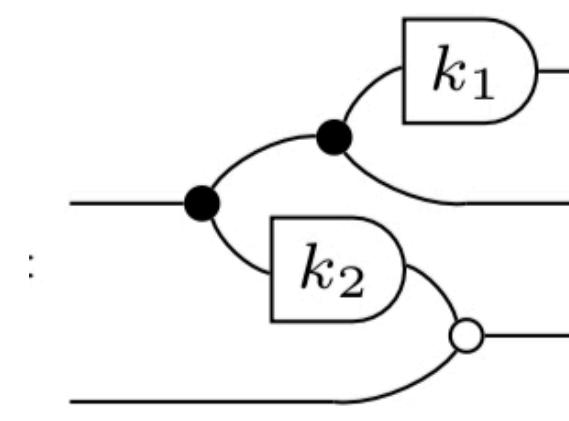
$$c ::= \bullet \mid \text{---}\bullet\text{---} \mid \text{---}k\text{---} \mid \text{---}\circ\text{---} \mid \circ\text{---} \mid \quad (1)$$

$$\bullet\text{---} \mid \text{---}\bullet\text{---} \mid \text{---}(k)\text{---} \mid \text{---}\circ\text{---} \mid \text{---}\circ\text{---} \mid \quad (2)$$

$$\text{---}\geq\text{---} \mid \quad (3)$$

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PDiag

3 → 4

Diagrammatic Polyhedral Algebra Semantics

$$[\cdot] : \text{PDiag} \rightarrow \text{Rel}_k$$

$$[\bullet \cap] = \{(x, \binom{x}{x}) \mid x \in k\}$$

Diagrammatic Polyhedral Algebra Semantics

$$[\![\cdot]\!]: \text{PDiag} \rightarrow \text{Rel}_k$$

$$[\![\bullet \cap]\!] = \{(x, \binom{x}{x}) \mid x \in k\} \quad [\![\cap \circ]\!] = \{(\binom{x}{y}, x+y) \mid x, y \in k\}$$

Diagrammatic Polyhedral Algebra Semantics

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Diagrammatic Polyhedral Algebra Semantics

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Diagrammatic Polyhedral Algebra Semantics

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$$[\![\geq]\!] = \{ (x, y) \mid x, y \in k, x \geq y \}$$

$$[\![\vdash]\!] = \{(\bullet, 1)\}$$

Diagrammatic Polyhedral Algebra Semantics

$$[\![\cdot]\!]: \text{PDiag} \rightarrow \text{Rel}_k$$

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$$[\![\text{---}\bullet]\!] = \{(x, \bullet) \mid x \in k\} \quad [\![\circ\text{---}]\!] = \{(\bullet, 0)\} \quad [\![\text{---}\square[k]\text{---}]\!] = \{(x, k \cdot x) \mid x \in k\}$$

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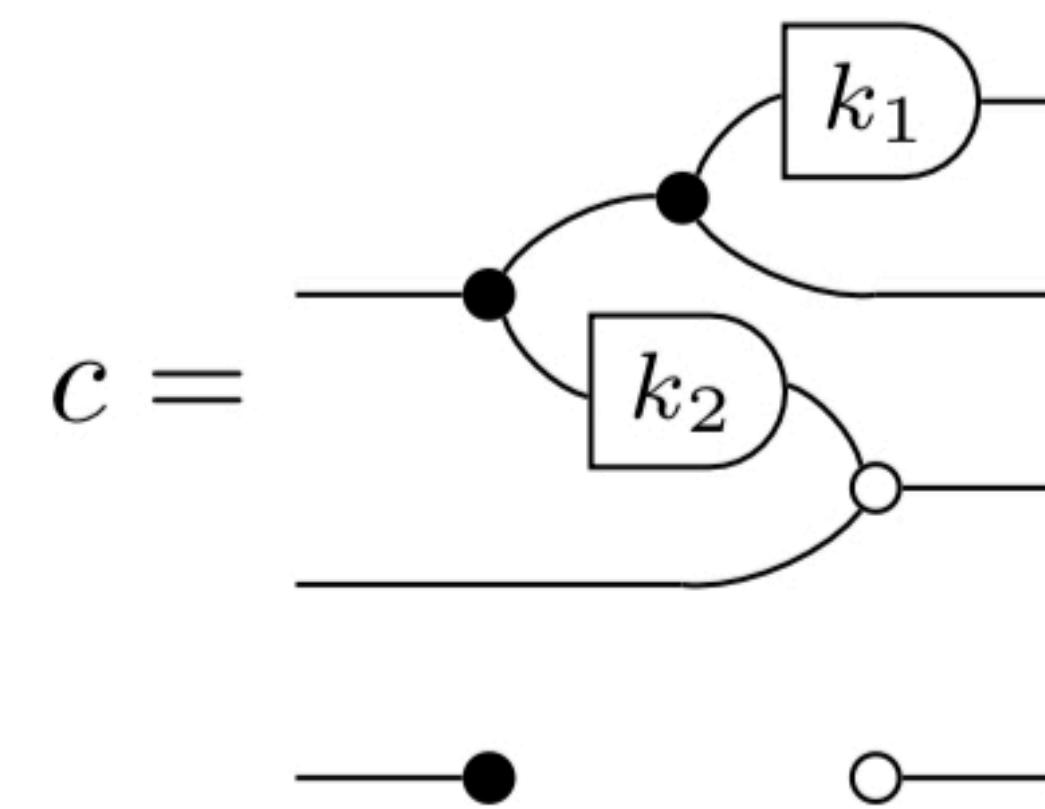
$$[\![\text{---}\trianglerighteq\text{---}]\!] = \{(x, y) \mid x, y \in k, x \geq y\}$$

$$[\![\vdash]\!] = \{(\bullet, 1)\}$$

$$[\![c ; d]\!] = \{(x, y) \mid \exists z. (x, z) \in [\![c]\!] \wedge (z, y) \in [\![d]\!]\}$$

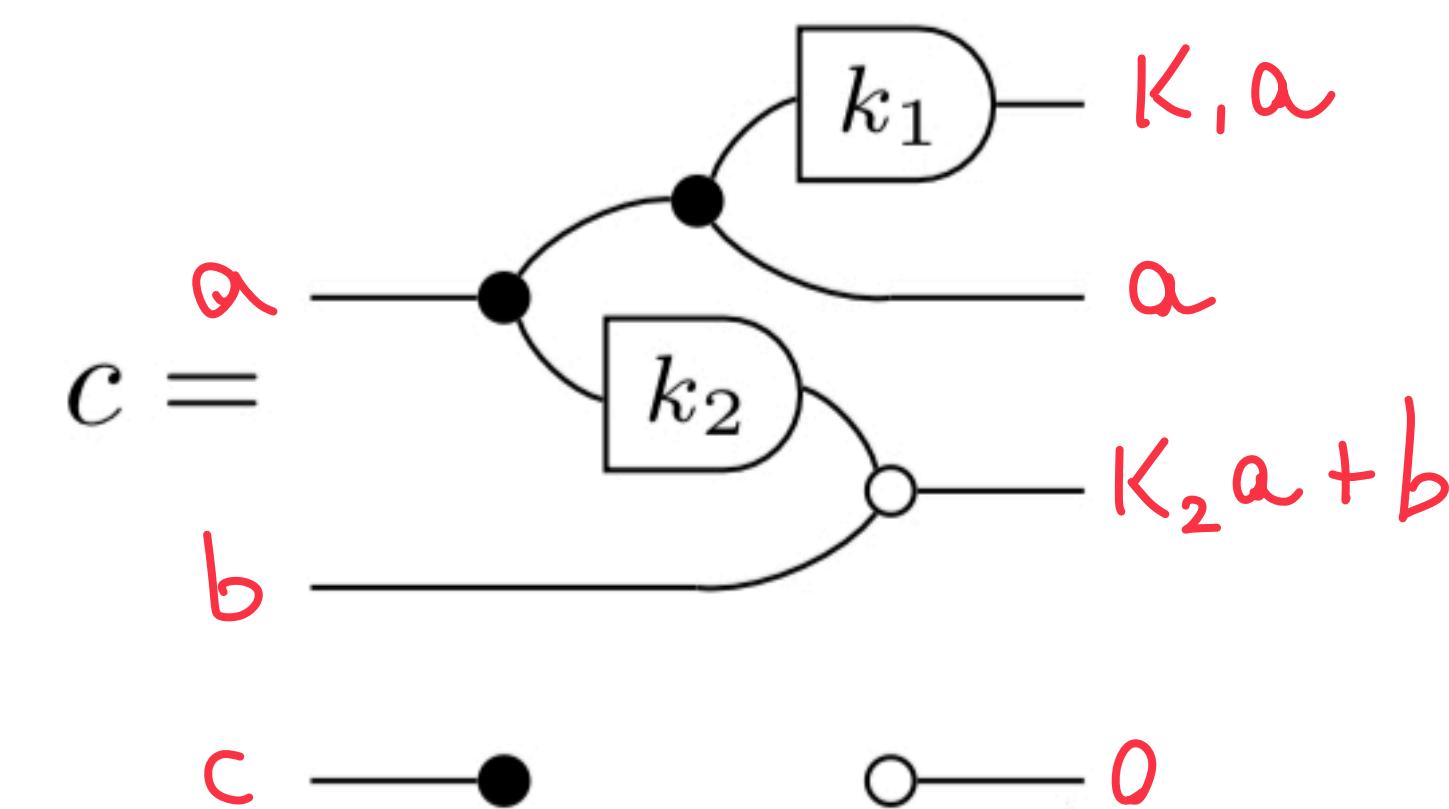
$$[\![c \oplus d]\!] = \{\left(\binom{x_1}{x_2}, \binom{y_1}{y_2}\right) \mid (x_1, y_1) \in [\![c]\!] \wedge (x_2, y_2) \in [\![d]\!]\}$$

Matrices



$$A = \begin{pmatrix} k_1 & 0 & 0 \\ 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

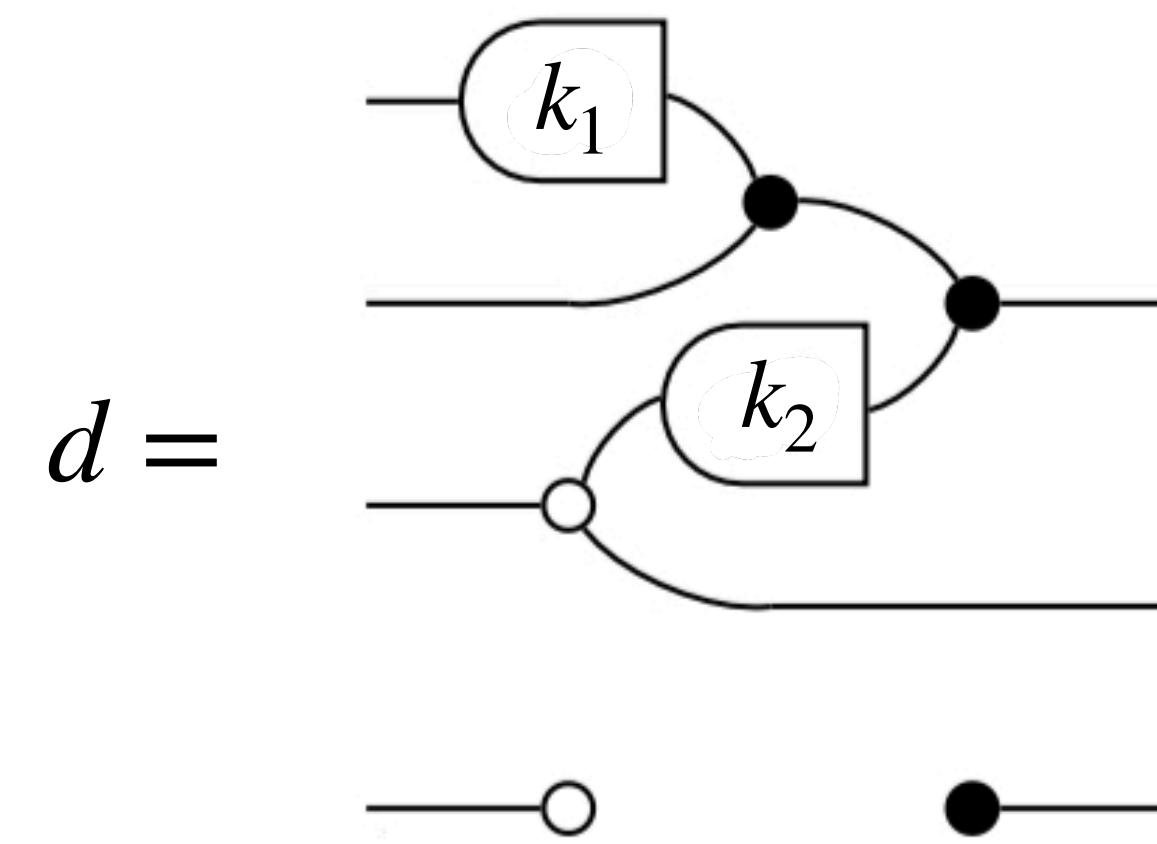
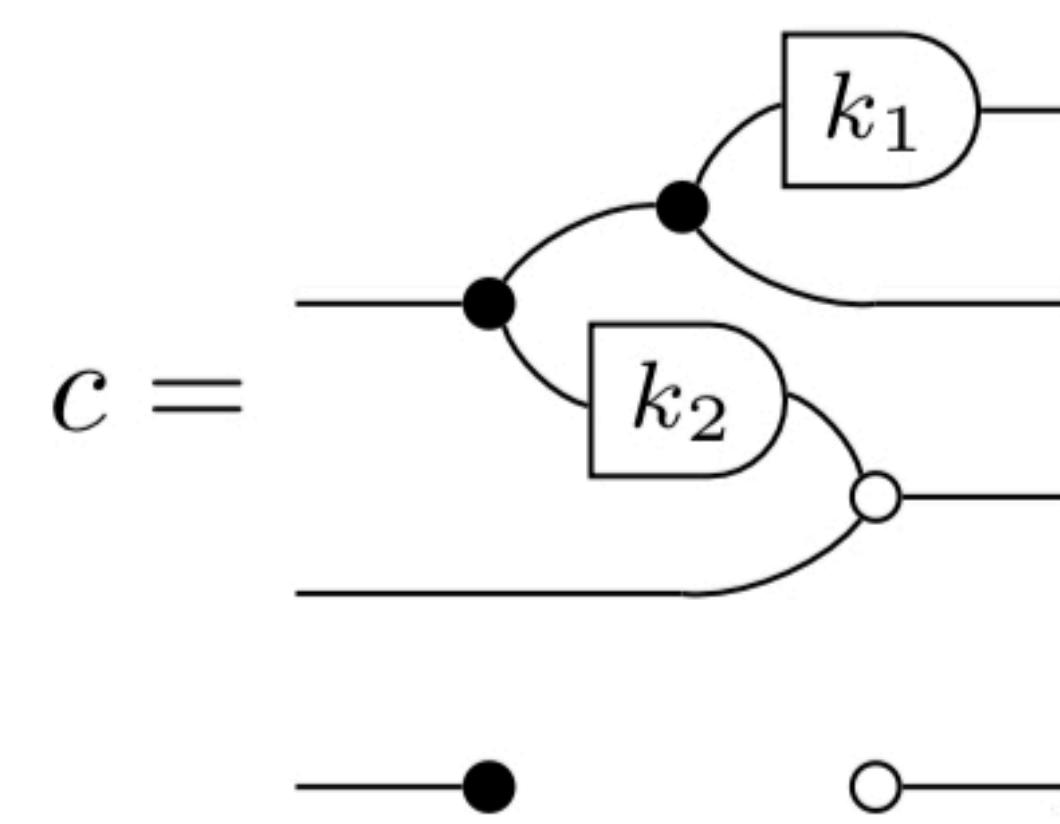
Matrices



$$A = \begin{pmatrix} k_1 & 0 & 0 \\ 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} K_1 \cdot a \\ a \\ K_2 \cdot a + b \\ 0 \end{pmatrix}$$

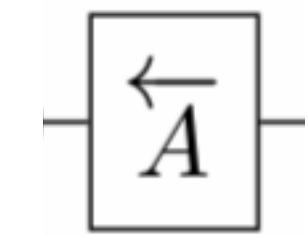
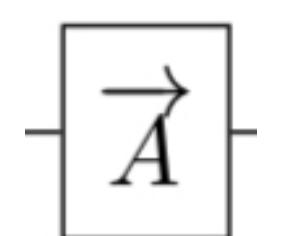
$$\llbracket C \rrbracket = \{(x, Ax) \mid x \in \mathbb{k}^3\}$$

Reversed matrices

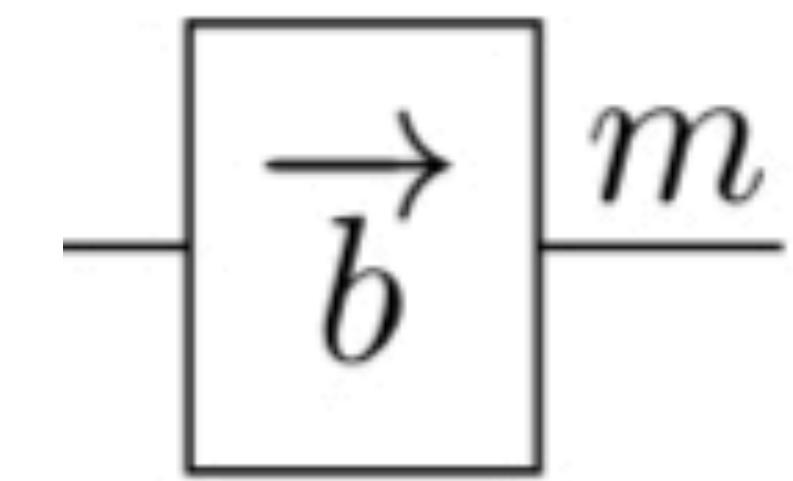


$$\llbracket c \rrbracket = \{(x, Ax) \mid x \in \mathbb{k}^3\}$$

$$\llbracket d \rrbracket = \{(Ax, x) \mid x \in \mathbb{k}^3\}$$

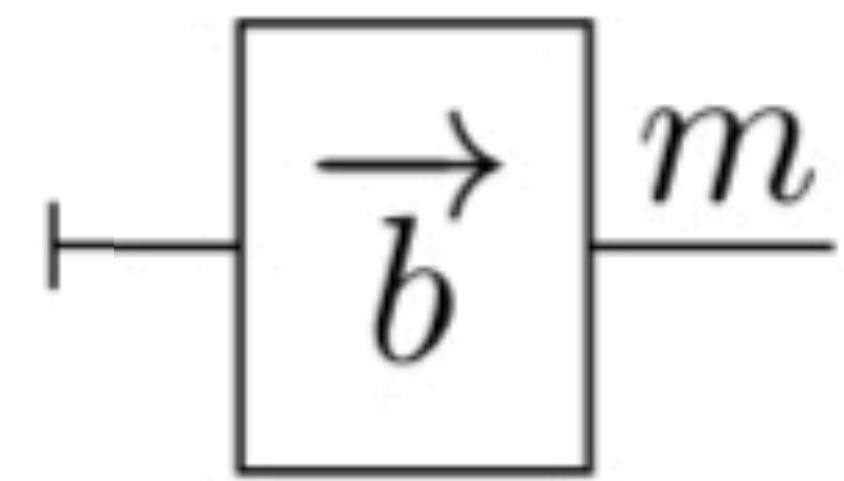


Vectors



$$\{(k, bk) \in \mathbf{k}^1 \times \mathbf{k}^m \mid k \in \mathbf{k}\}$$

Vectors



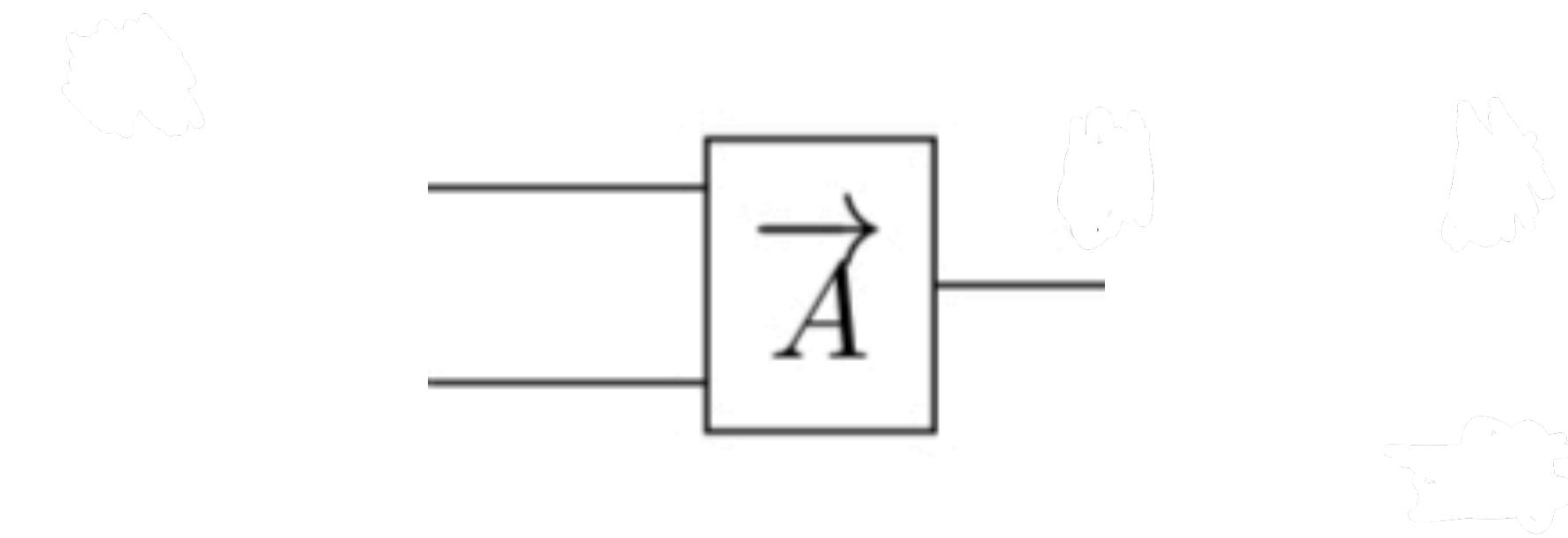
$$\{(\bullet, b) \in \mathbb{k}^0 \times \mathbb{k}^m\}$$

Polyhedral cones

$$\{(x, y) \in \mathbb{k}^n \times \mathbb{k}^m \mid A \begin{pmatrix} x \\ y \end{pmatrix} \geq 0\}$$

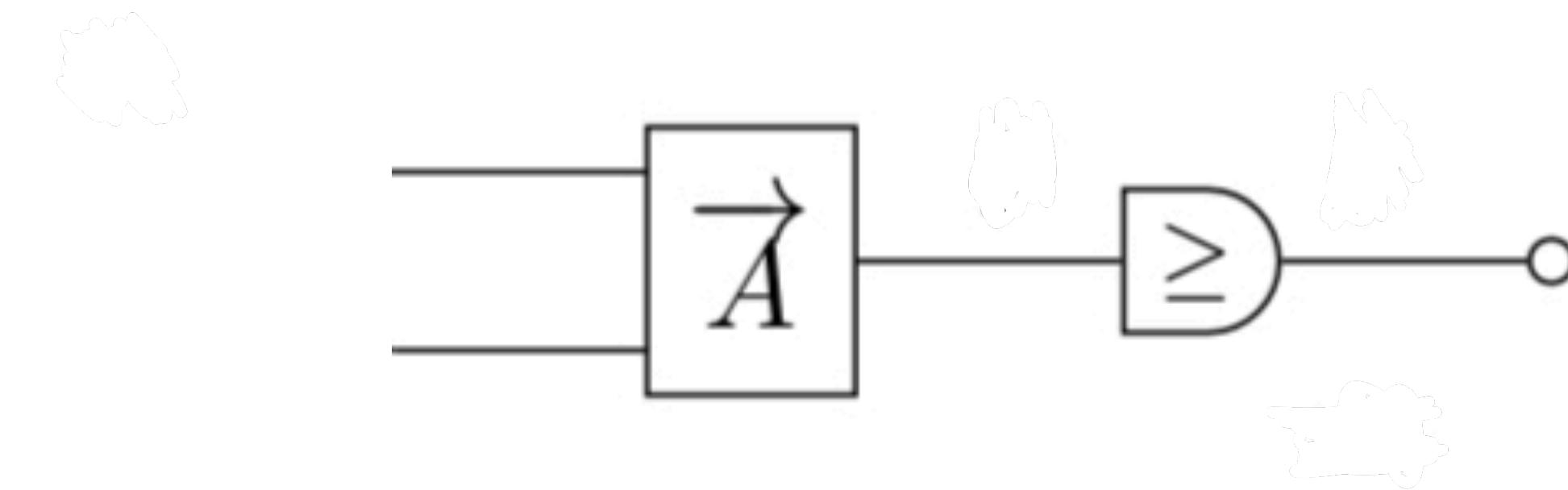
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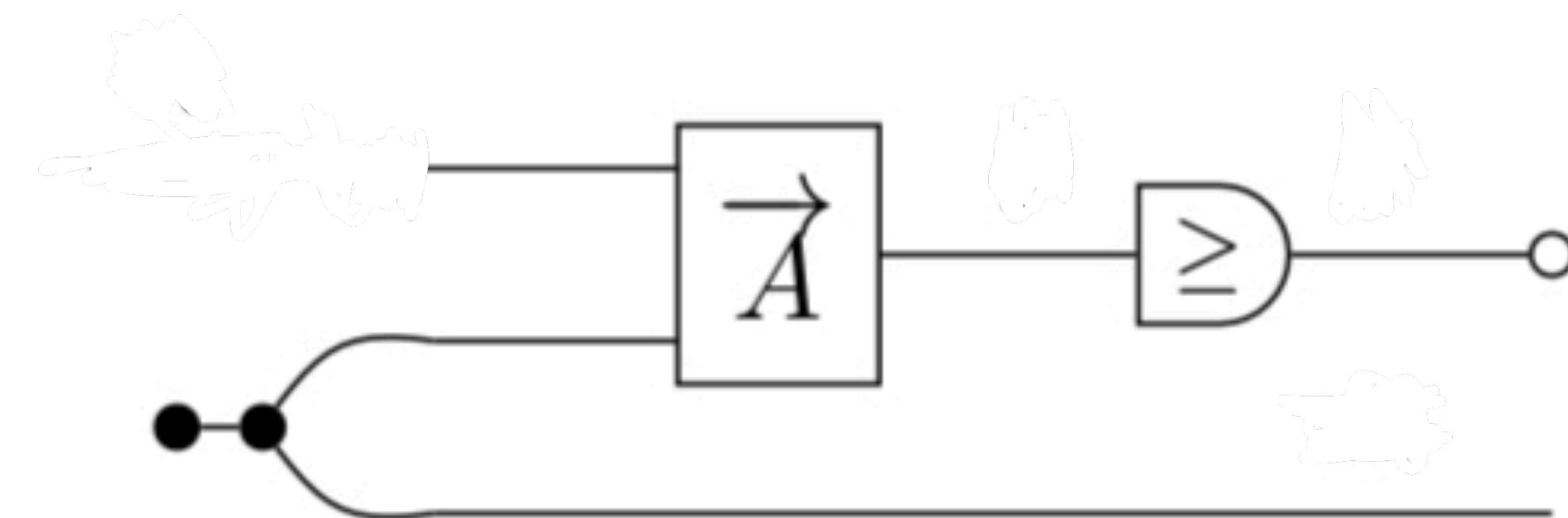
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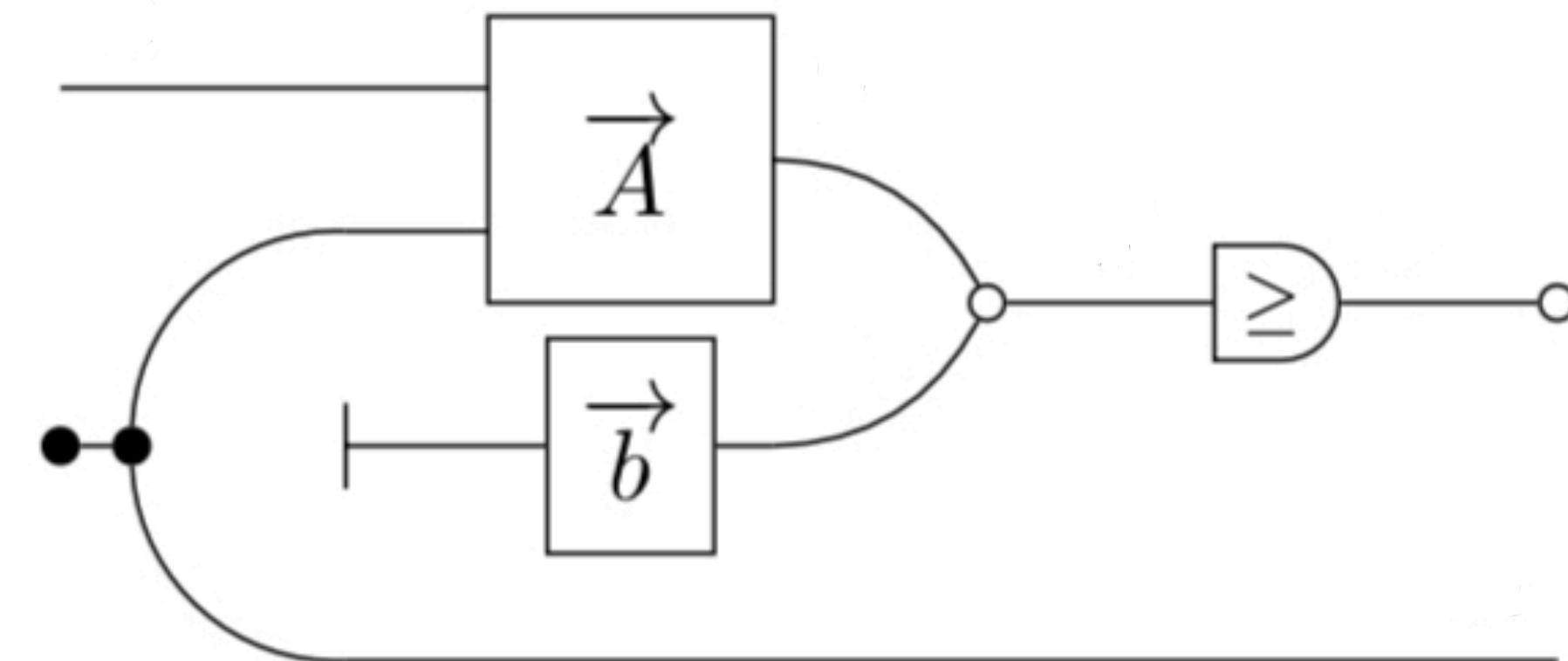
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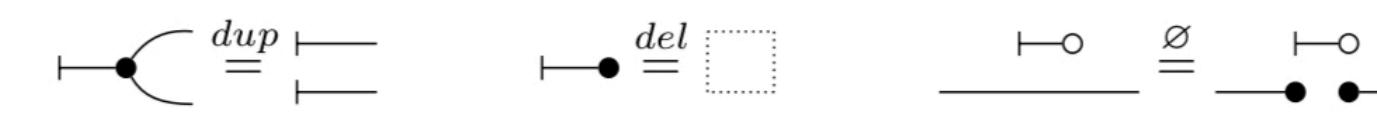
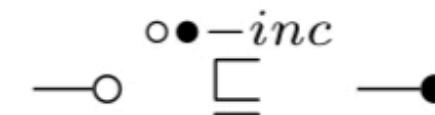
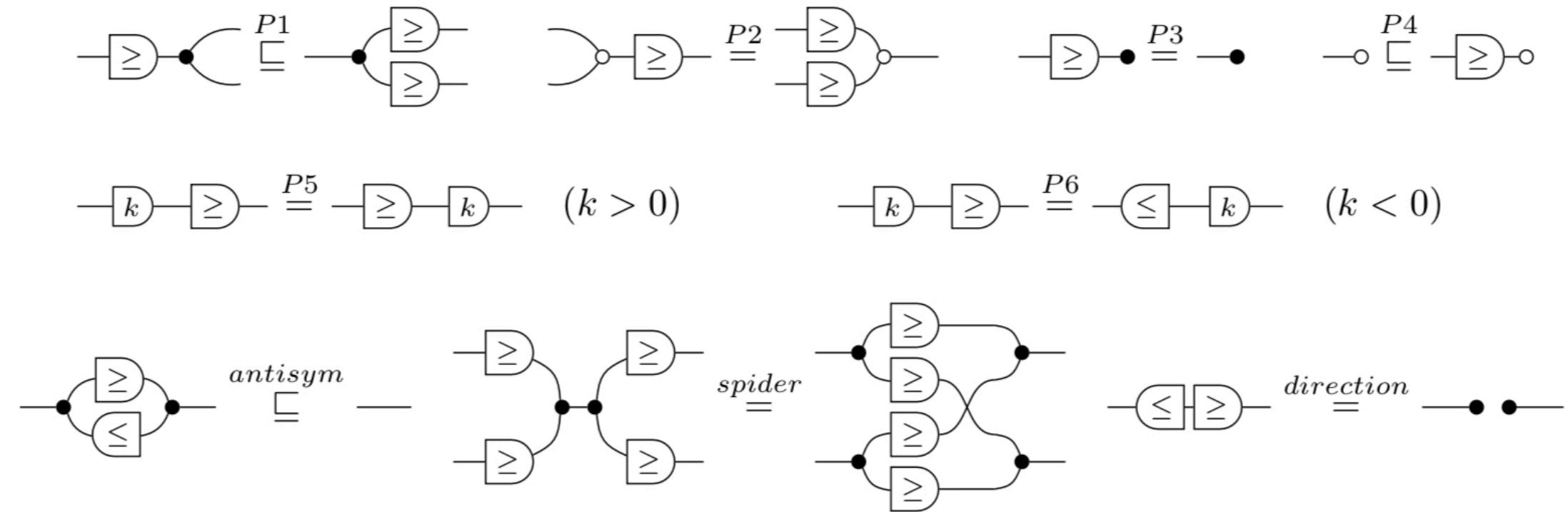
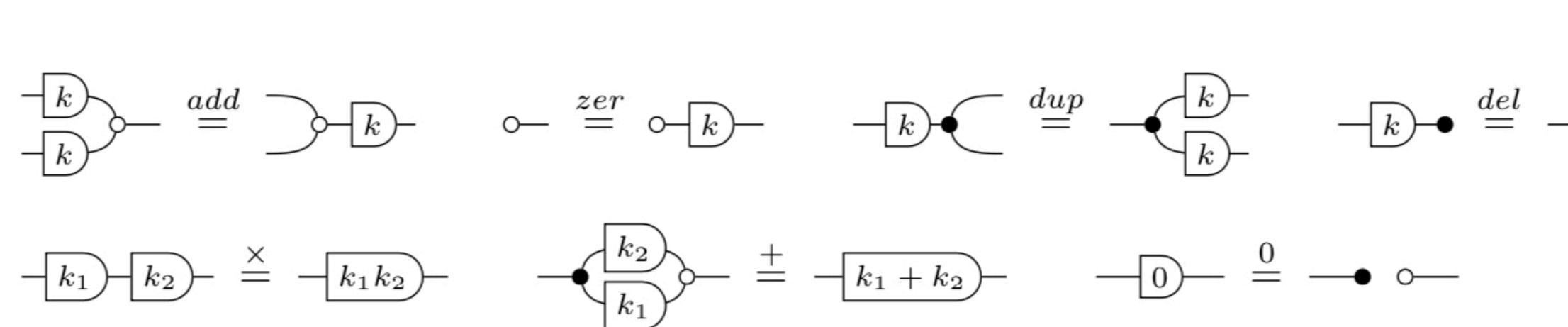
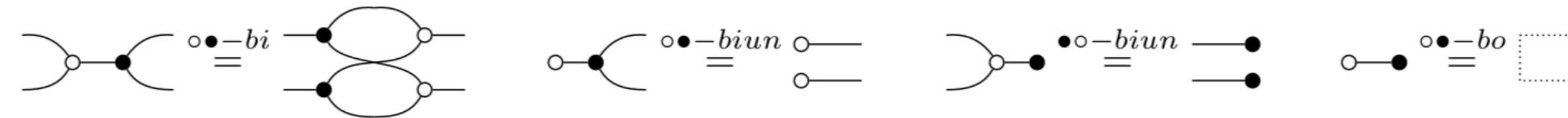
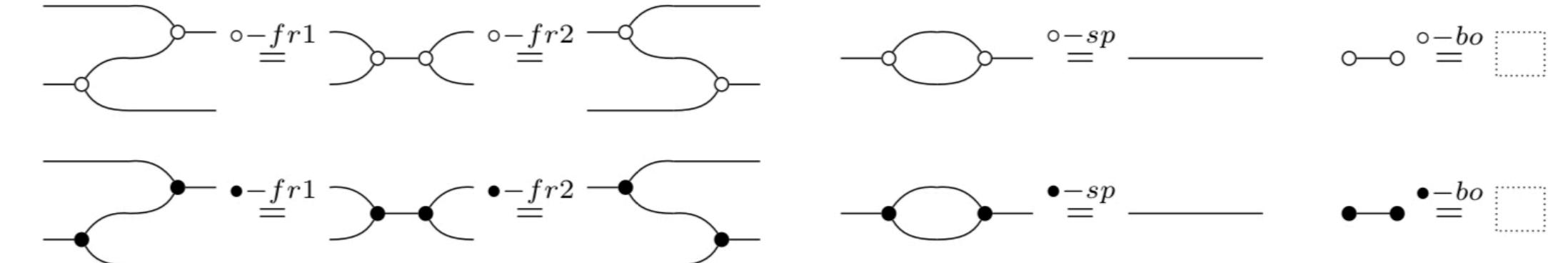
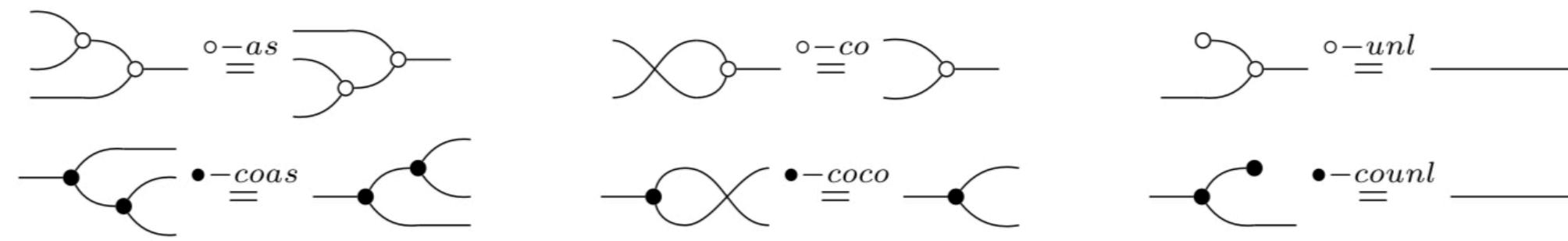


Polyhedra

$$\{ (x, y) \in \mathbb{k}^n \times \mathbb{k}^m \mid A \begin{pmatrix} x \\ y \end{pmatrix} + b \geq 0 \}$$



(Sound and complete) axiomatisation

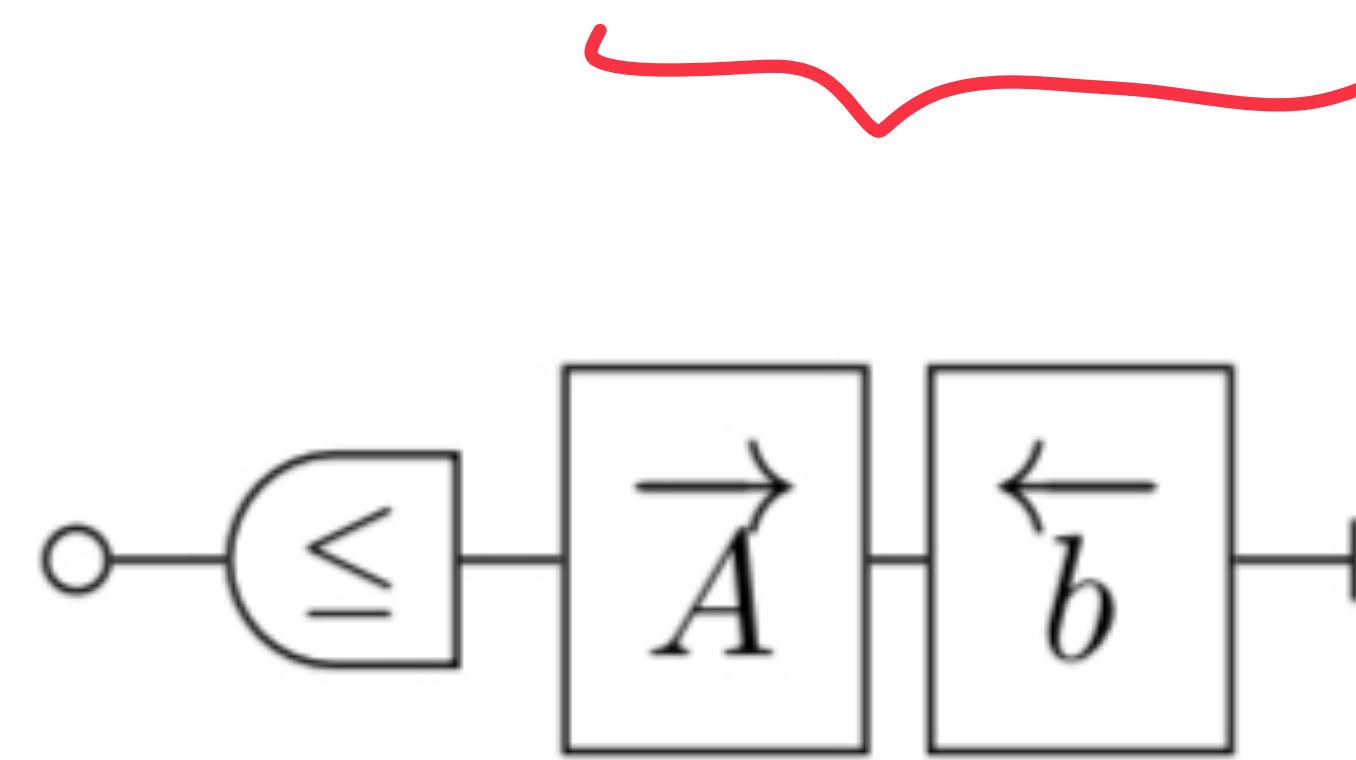


Farkas' Lemma

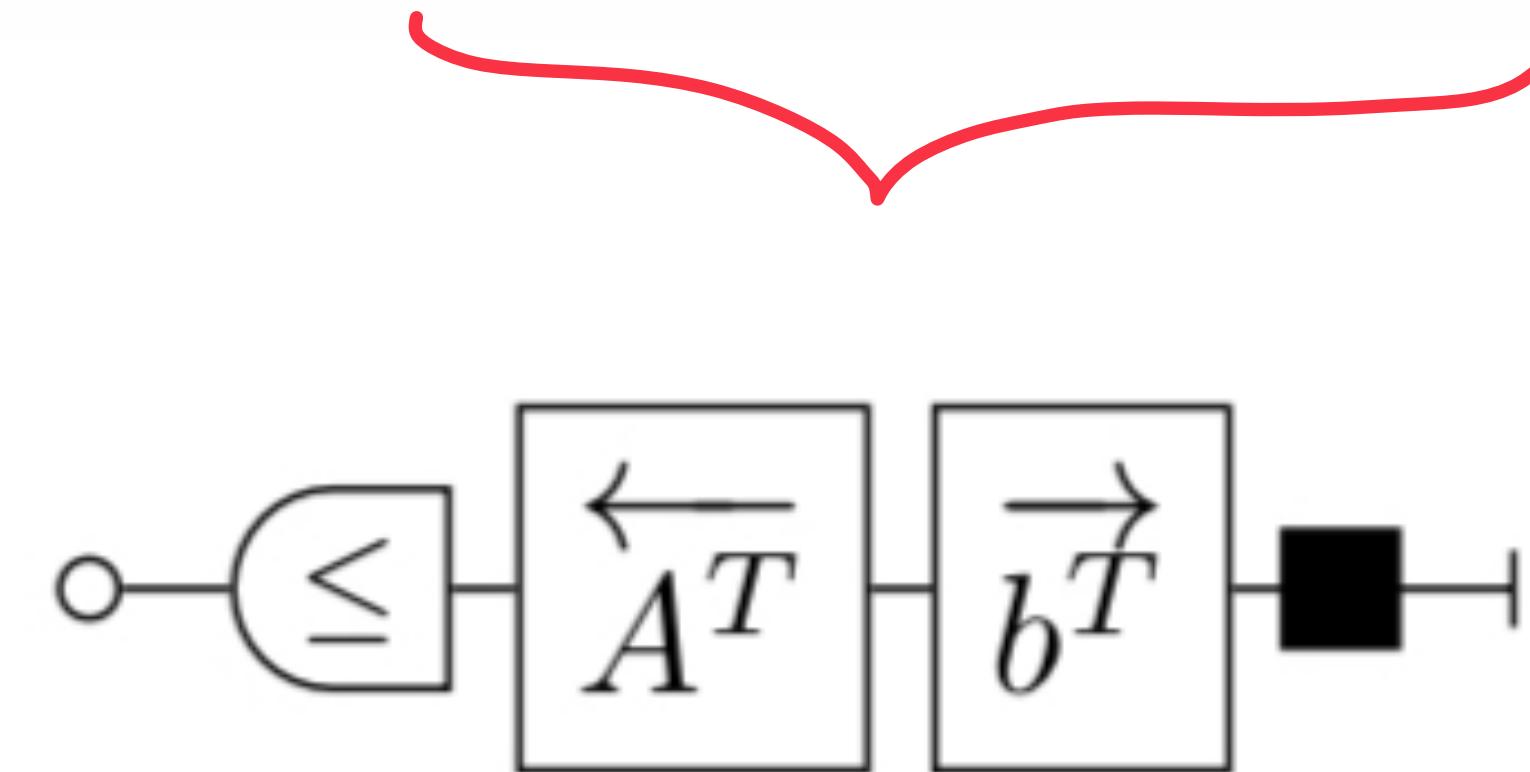
- (a) $\exists x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$ (b) $\exists y \in \mathbb{k}^m$ s.t. $A^T y \geq 0$ and $b^T y = -1$

Farkas' Lemma

(a) $\exists x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$

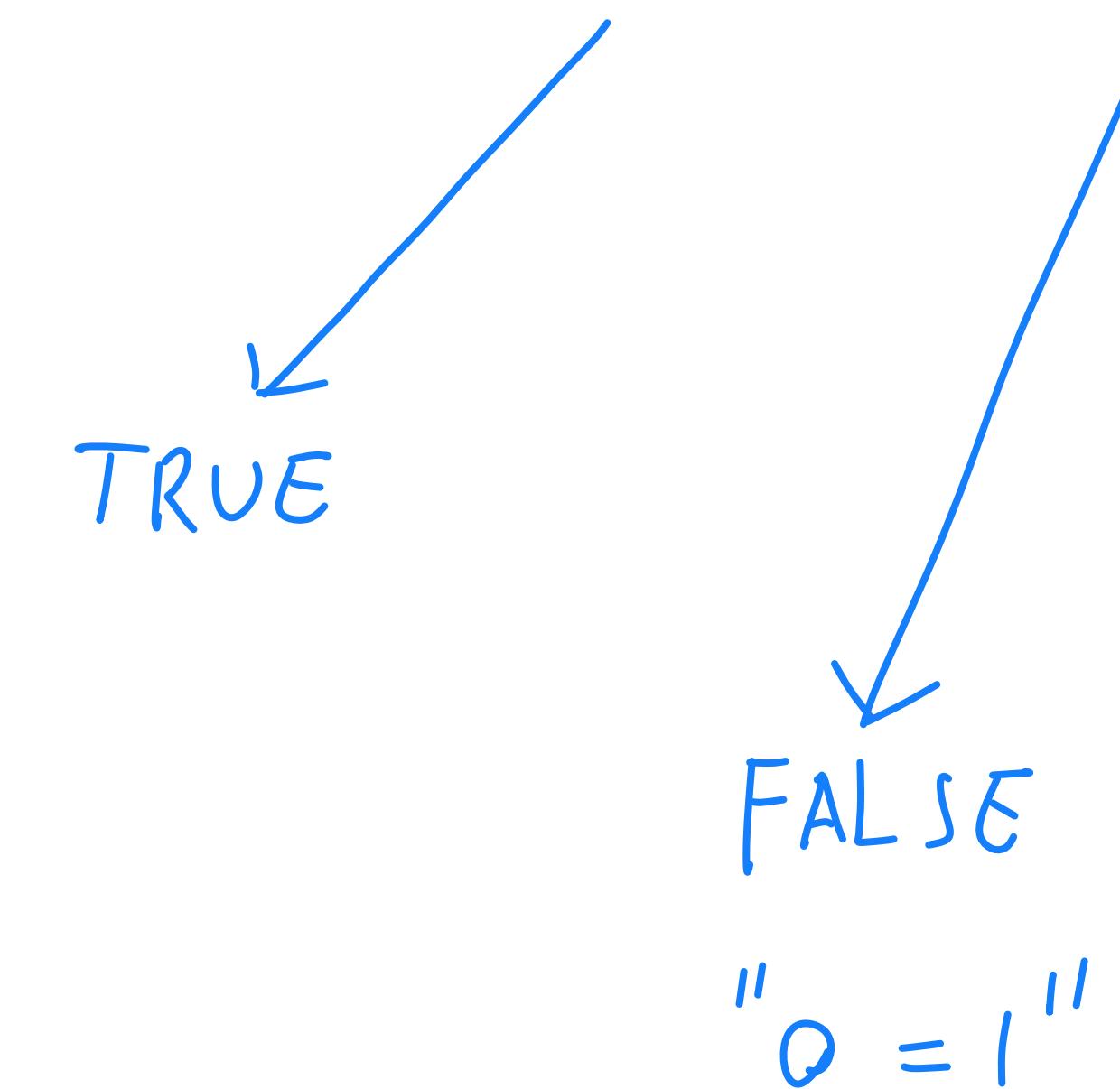


(b) $\exists y \in \mathbb{k}^m$ s.t. $A^T y \geq 0$ and $b^T y = -1$



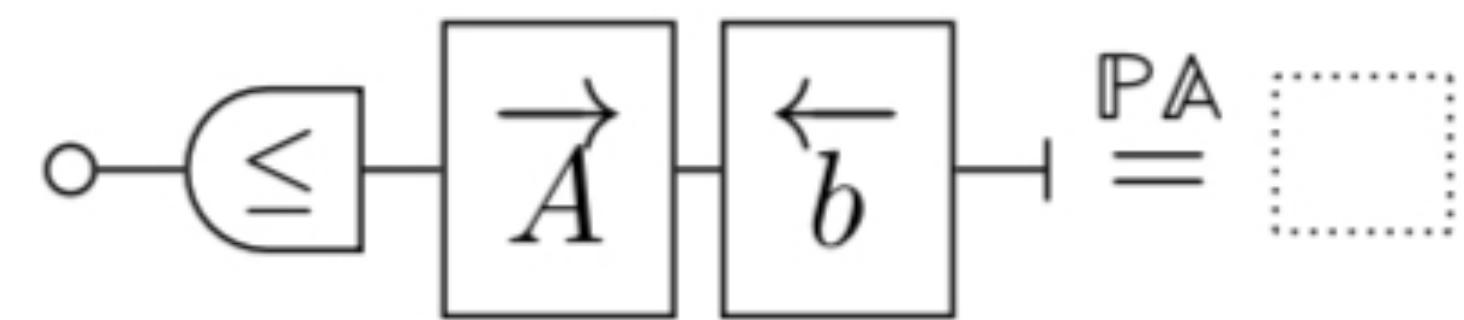
Diagrams $0 \rightarrow 0$

► **Lemma 17** For any diagram $c: 0 \rightarrow 0$ of PDiag, either $c \stackrel{\mathbb{P}\mathbb{A}}{=} \square$ or $c \stackrel{\mathbb{P}\mathbb{A}}{=} \circ \dashv$

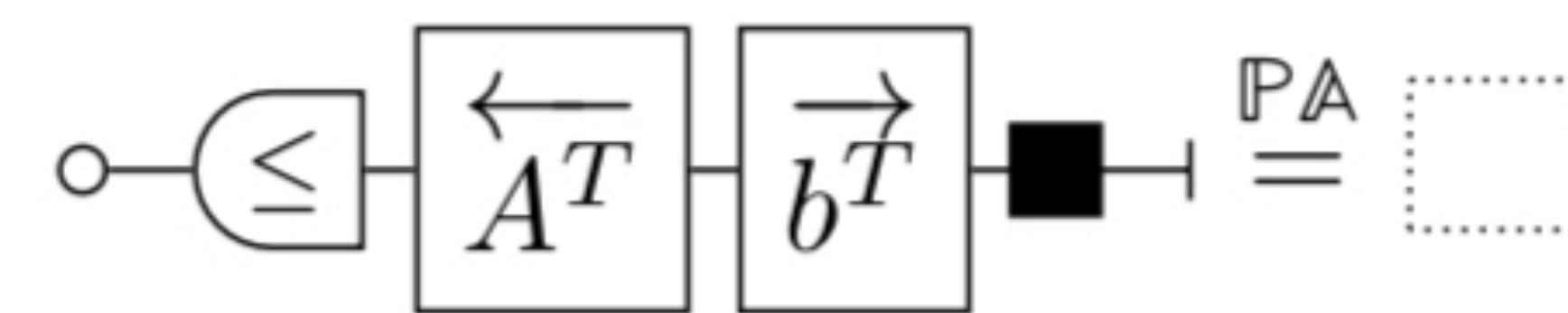


Farkas' Lemma

(a) $\exists x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$



(b) $\exists y \in \mathbb{k}^m$ s.t. $A^T y \geq 0$ and $b^T y = -1$

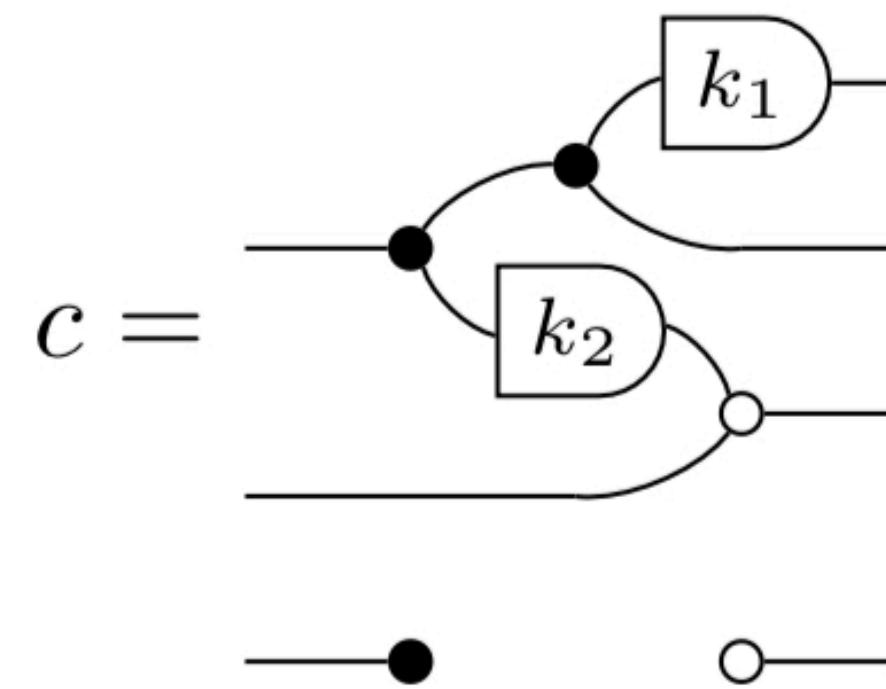


The polar operator

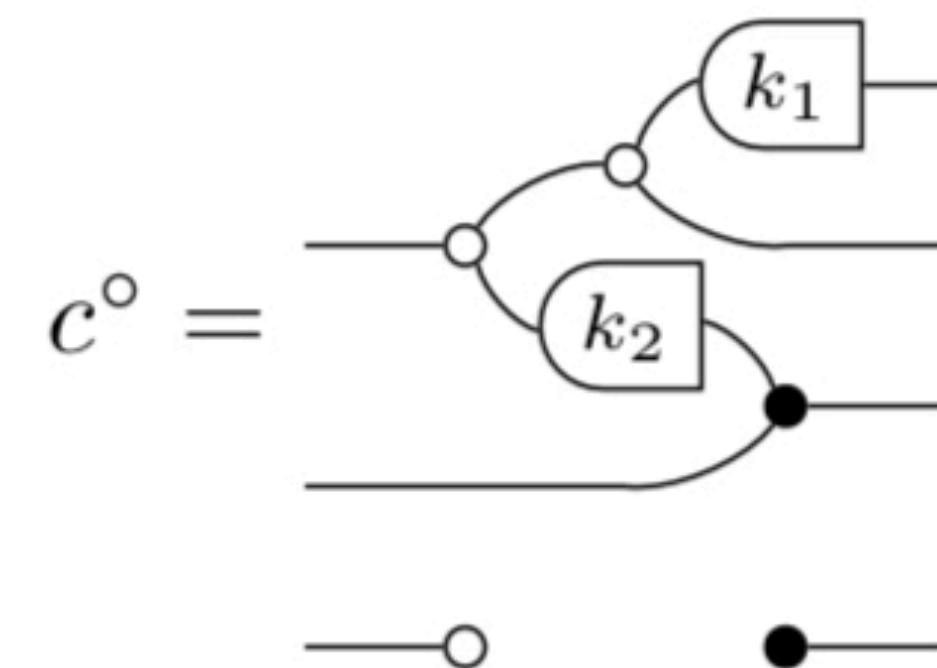
► **Definition 10.** The prop morphism $\cdot^\circ: \text{PCDiag} \rightarrow \text{PCDiag}$ is inductively defined as:

$$\begin{array}{lllll} \bullet^\circ = \circ & \bullet^\circ = \circ & \square[k]^\circ = \square[k] & \circ^\circ = \bullet & \circ^\circ = \bullet \\ \circ^\circ = \bullet & \circ^\circ = \bullet & \square[k]^\circ = \square[k] & \bullet^\circ = \circ & \bullet^\circ = \circ \\ \square^\circ = \square & \circ^\circ = \circ & (c \oplus d)^\circ = c^\circ \oplus d^\circ & (c; d)^\circ = c^\circ; d^\circ & \times^\circ = \times \\ & & & & \\ & & \square[\geq]^\circ = \bullet \square[\geq] \circ & & \end{array}$$

The polar operator (on matrices)



$$A = \begin{pmatrix} k_1 & 0 & 0 \\ 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$A^T = \begin{pmatrix} k_1 & 1 & k_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Farkas' Lemma (diagrammatic proof)

► **Lemma 20** (Farkas lemma). Let $\vec{A} : n \rightarrow m$ be a diagram in $\text{MDiag}^{\rightarrow}$ and $\overleftarrow{b} : m \rightarrow 1$ in $\text{MDiag}^{\leftarrow}$, then exactly one of the following two equations holds:

$$(a) \quad o - \circ - \begin{array}{|c|}\hline \leq \\ \hline\end{array} - \boxed{\vec{A}} - \boxed{\overleftarrow{b}} - \xrightarrow{\text{PA}} = \square \quad (b) \quad o - \circ - \begin{array}{|c|}\hline \leq \\ \hline\end{array} - \boxed{\overleftarrow{A^T}} - \boxed{\overrightarrow{b^T}} - \blacksquare - \xrightarrow{\text{PA}} = \square$$

Farkas' Lemma (diagrammatic proof)

► **Lemma 20** (Farkas lemma). *Let $\vec{A}: n \rightarrow m$ be a diagram in $\text{MDiag}^{\rightarrow}$ and $\overleftarrow{b}: m \rightarrow 1$ in $\text{MDiag}^{\leftarrow}$, then exactly one of the following two equations holds:*

$$(a) \quad o -\!\!\!-\! \leq \!-\! \vec{A} \!-\! \overleftarrow{b} \!-\! \vdash \stackrel{\mathbb{P}\mathbb{A}}{=} \square \quad (b) \quad o -\!\!\!-\! \leq \!-\! \overleftarrow{A^T} \!-\! \overrightarrow{b^T} \!-\! \blacksquare \!-\! \vdash \stackrel{\mathbb{P}\mathbb{A}}{=} \square$$

► **Lemma 18** (Lemma of the alternatives). *Let $c: 0 \rightarrow 1$ be a diagram in PCDiag . Then exactly one of the following two equations holds:*

$$(a) \quad c; \dashv \stackrel{\mathbb{P}\mathbb{A}}{=} \square \quad (b) \quad c^\circ; \dashv \stackrel{\mathbb{P}\mathbb{A}}{=} \square.$$

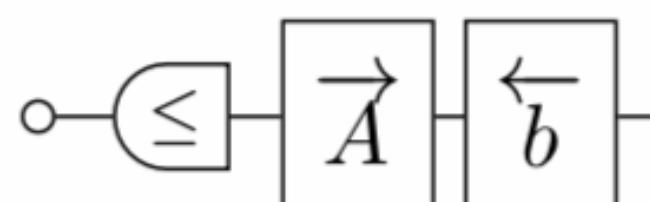
Farkas' Lemma (diagrammatic proof)

► **Lemma 20** (Farkas lemma). Let $\vec{A} : n \rightarrow m$ be a diagram in $\text{MDiag}^{\rightarrow}$ and $\overleftarrow{b} : m \rightarrow 1$ in $\text{MDiag}^{\leftarrow}$, then exactly one of the following two equations holds:

$$(a) \quad o \circ \begin{array}{c} \sqsubseteq \\[-1ex] \square \end{array} \xrightarrow{\vec{A}} \xrightarrow{\overleftarrow{b}} \xrightarrow{\text{PA}} \square \quad (b) \quad o \circ \begin{array}{c} \sqsubseteq \\[-1ex] \square \end{array} \xrightarrow{\overleftarrow{A^T}} \xrightarrow{\overrightarrow{b^T}} \blacksquare \xrightarrow{\text{PA}} \square$$

► **Lemma 18** (Lemma of the alternatives). Let $c : 0 \rightarrow 1$ be a diagram in PCDiag . Then exactly one of the following two equations holds:

$$(a) \quad c; \dashv \xrightarrow{\text{PA}} \square \quad (b) \quad c^\circ; \dashv \xrightarrow{\text{PA}} \square.$$



Farkas' Lemma (diagrammatic proof)

► **Lemma 20** (Farkas lemma). Let $\vec{A} : n \rightarrow m$ be a diagram in $\text{MDiag}^{\rightarrow}$ and $\overleftarrow{b} : m \rightarrow 1$ in $\text{MDiag}^{\leftarrow}$, then exactly one of the following two equations holds:

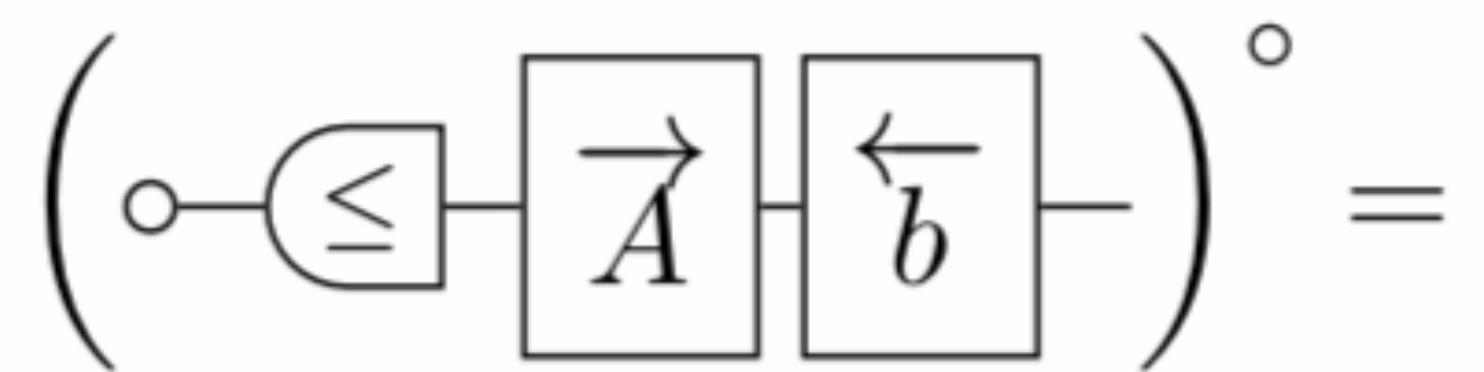
$$(a) \quad o \circ \begin{array}{c} \sqsubseteq \\ \square \end{array} \xrightarrow{\vec{A}} \begin{array}{c} \overleftarrow{b} \\ \square \end{array} \xrightarrow{\text{PA}} \square \quad (b) \quad o \circ \begin{array}{c} \sqsubseteq \\ \square \end{array} \xrightarrow{\overleftarrow{A}^T} \begin{array}{c} \overrightarrow{b}^T \\ \blacksquare \end{array} \xrightarrow{\text{PA}} \square$$

► **Lemma 18** (Lemma of the alternatives). Let $c : 0 \rightarrow 1$ be a diagram in PCDiag . Then exactly one of the following two equations holds:

$$(a) \quad c; \xrightarrow{\text{PA}} \square \quad (b) \quad c^\circ; \xrightarrow{\text{PA}} \square.$$

↓
 $\circ \circ \begin{array}{c} \sqsubseteq \\ \square \end{array} \xrightarrow{\vec{A}} \begin{array}{c} \overleftarrow{b} \\ \square \end{array}$ $\circ \circ \begin{array}{c} \sqsubseteq \\ \square \end{array} \xrightarrow{\overleftarrow{A}^T} \begin{array}{c} \overrightarrow{b}^T \\ \blacksquare \end{array}$

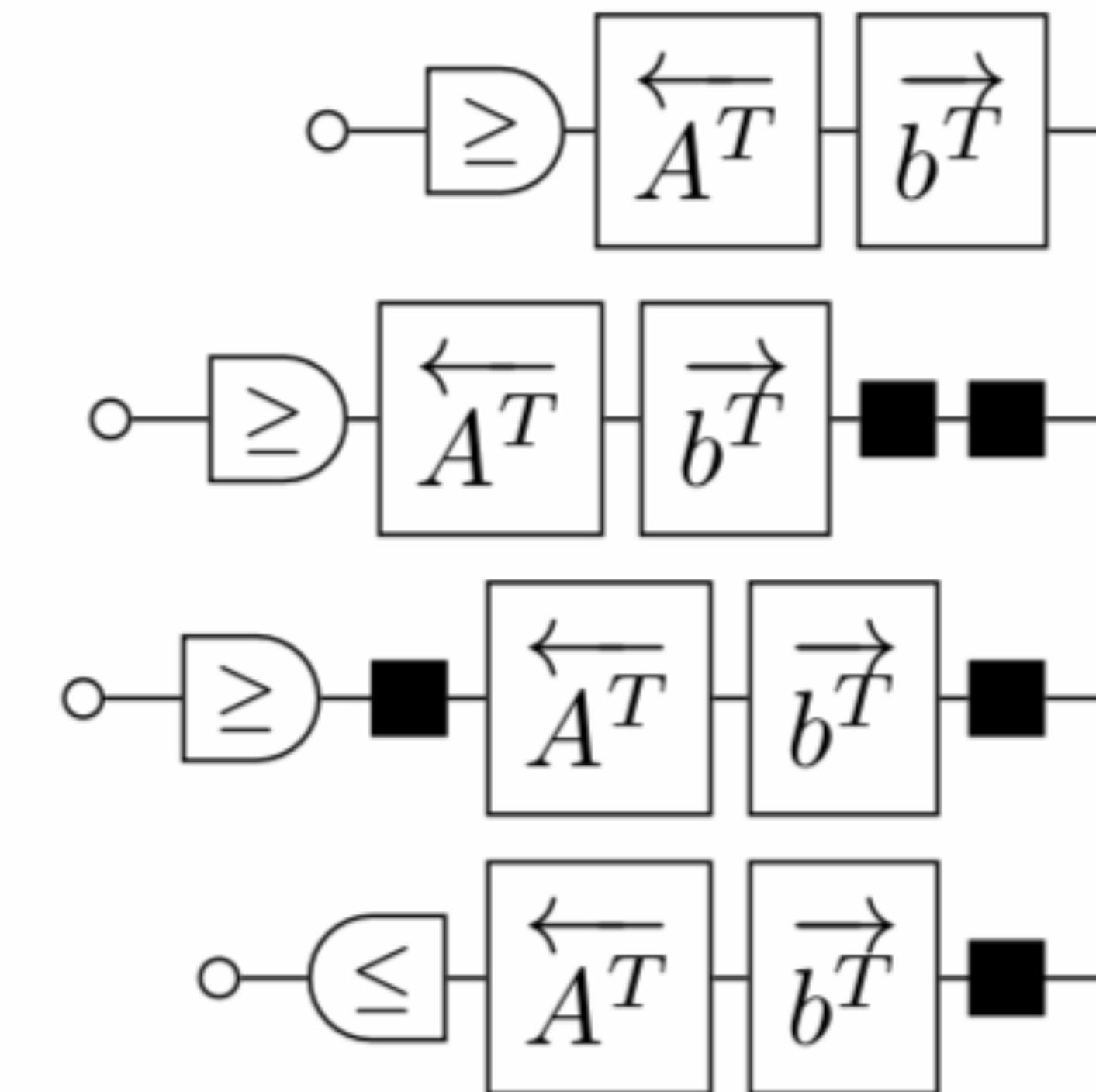
Farkas' Lemma (diagrammatic proof)



$\text{PA} \equiv$

$\text{PA} \equiv$

$\text{PA} \equiv$



Linear programming duality

$$(P) := \max\{cx \mid Ax \leq b, x \geq 0\}$$

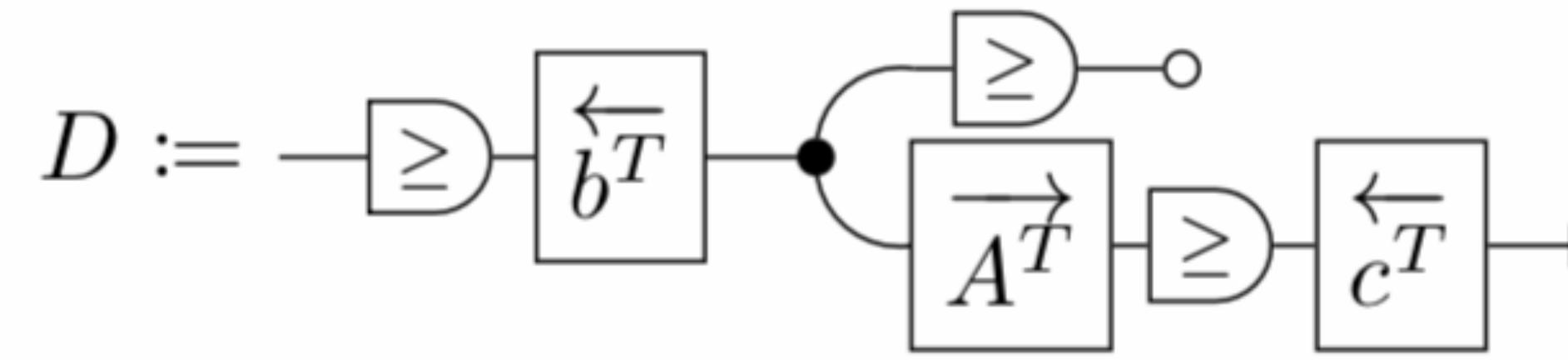
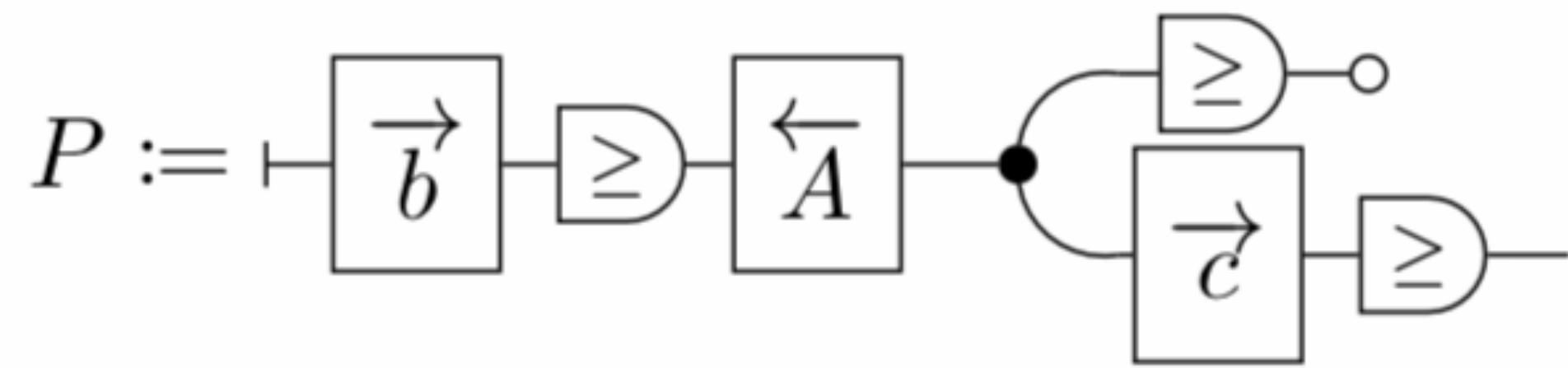
$$(D) := \min\{b^T y \mid A^T y \geq c^T \text{ and } y \geq 0\}$$

(P)	(D)	bounded	unbounded	unfeasible
bounded		✓		
unbounded				✓
unfeasible			✓	✓

Linear programming duality

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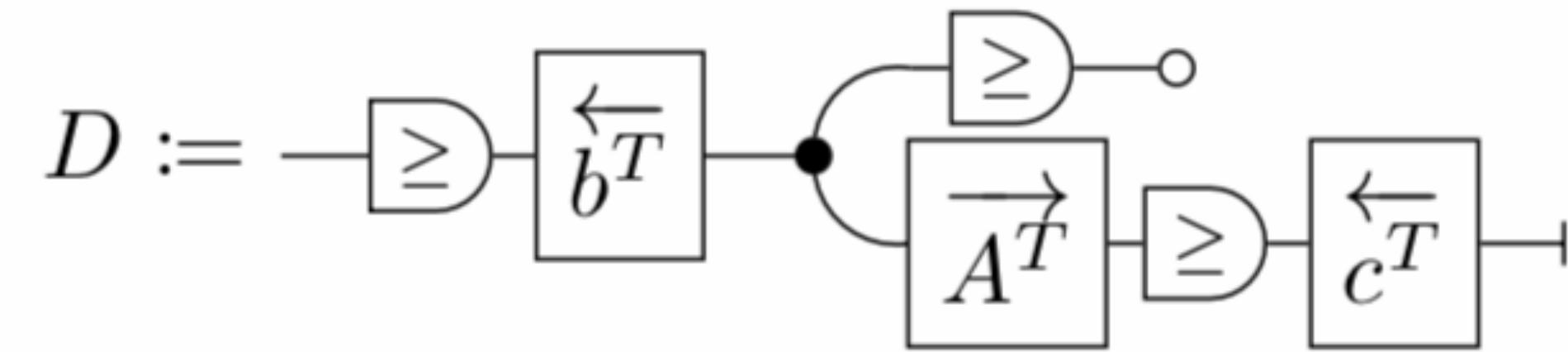
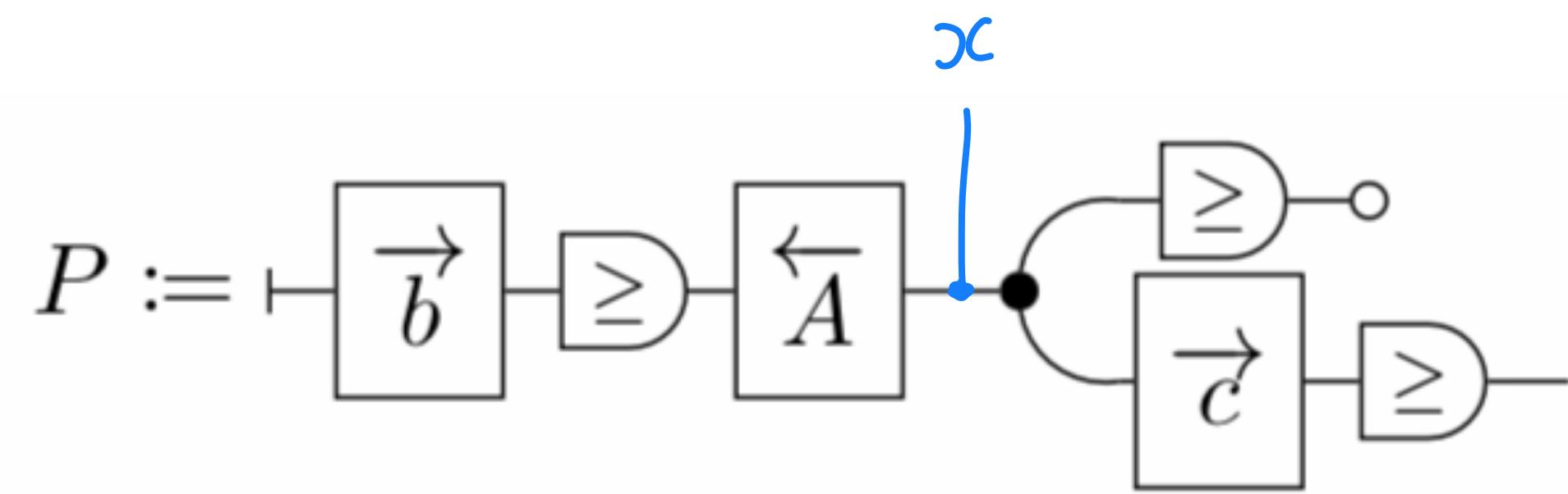
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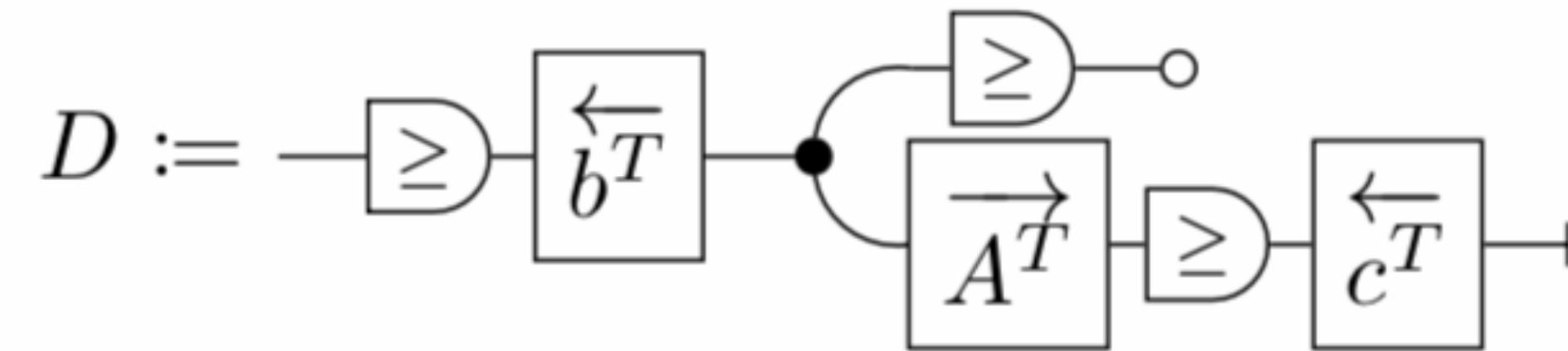
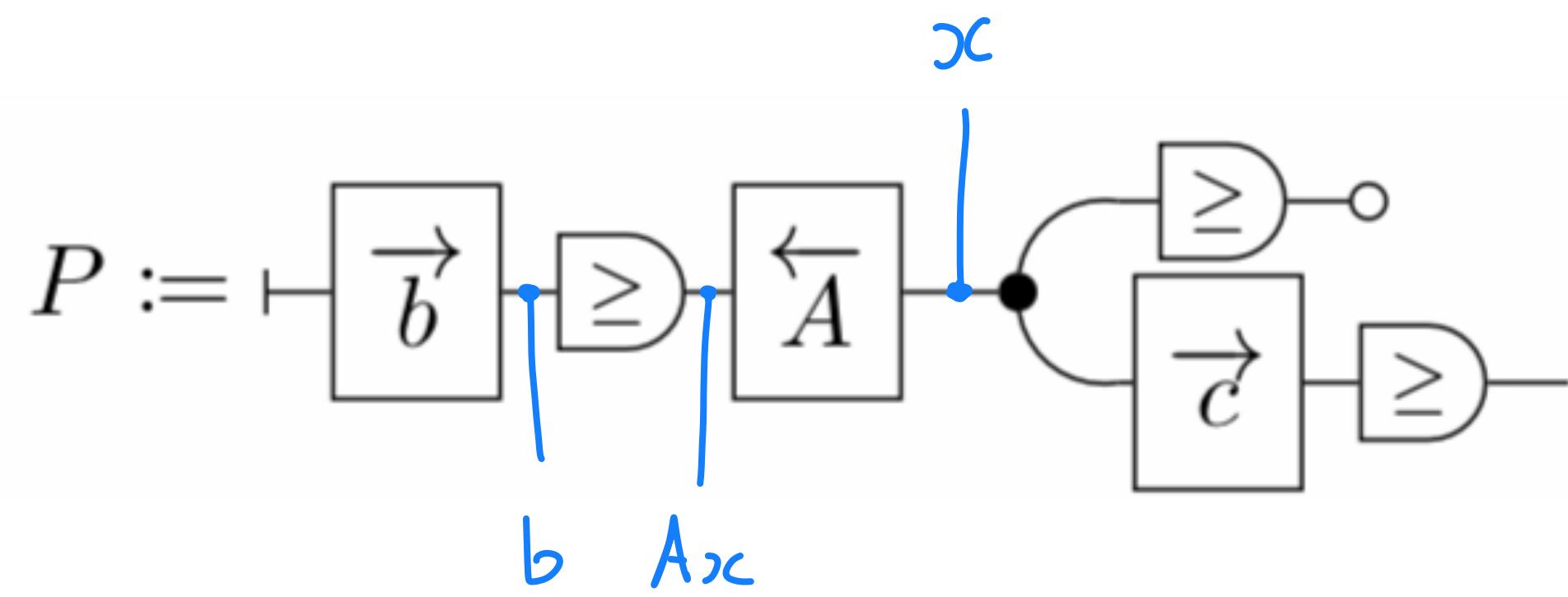
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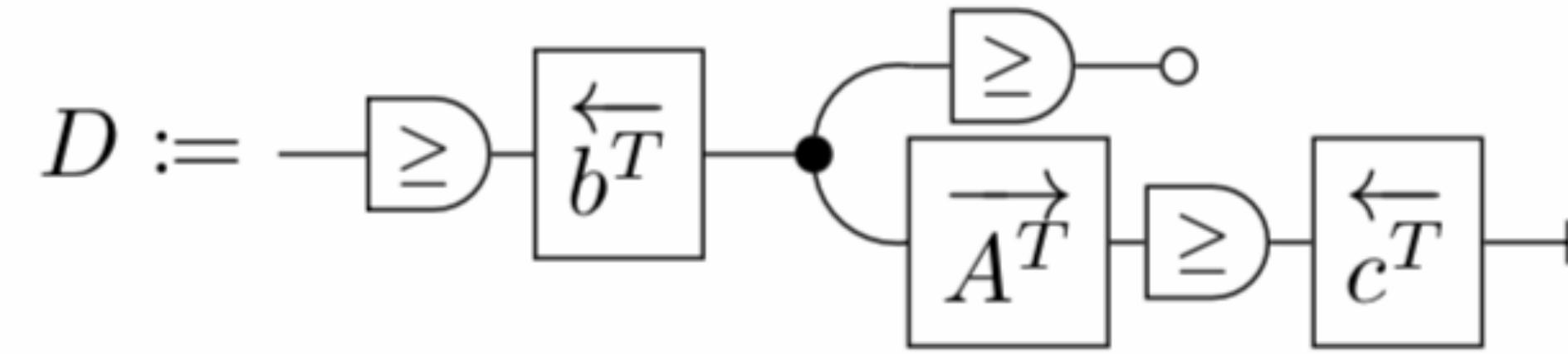
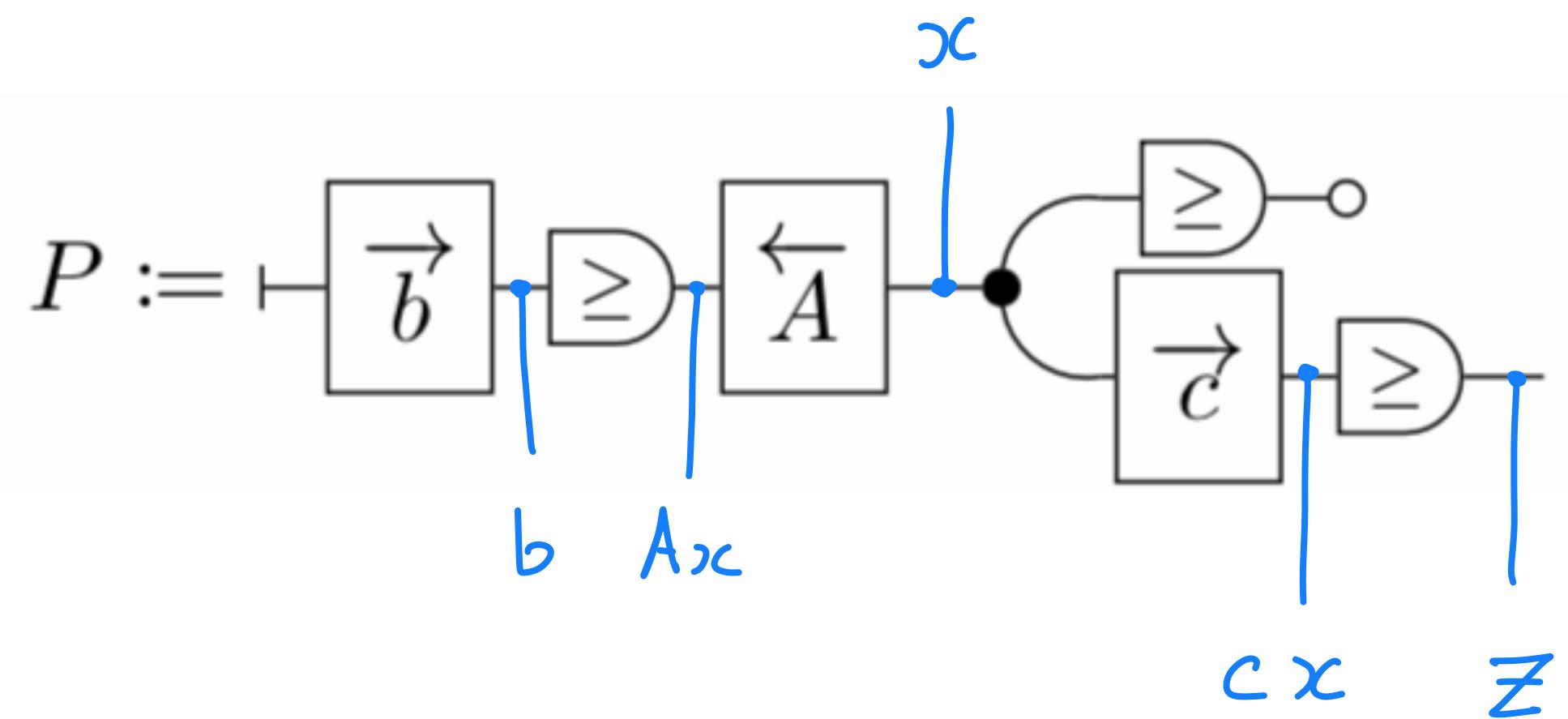
$$(D) := \min\{b^T y \mid A^T y \geq c^T \text{ and } y \geq 0\}$$



Linear programming duality

$$(P) := \max\{cx \mid Ax \leq b, x \geq 0\}$$

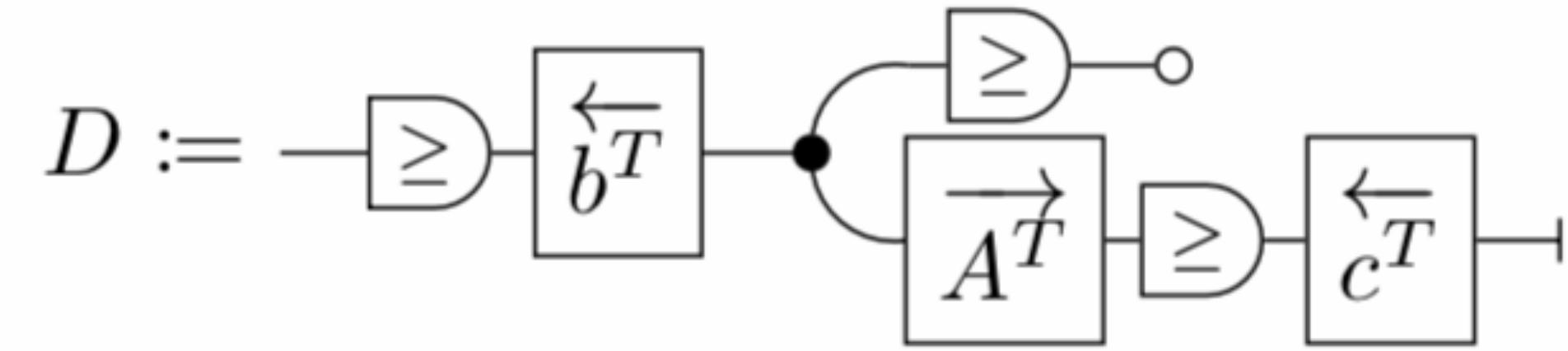
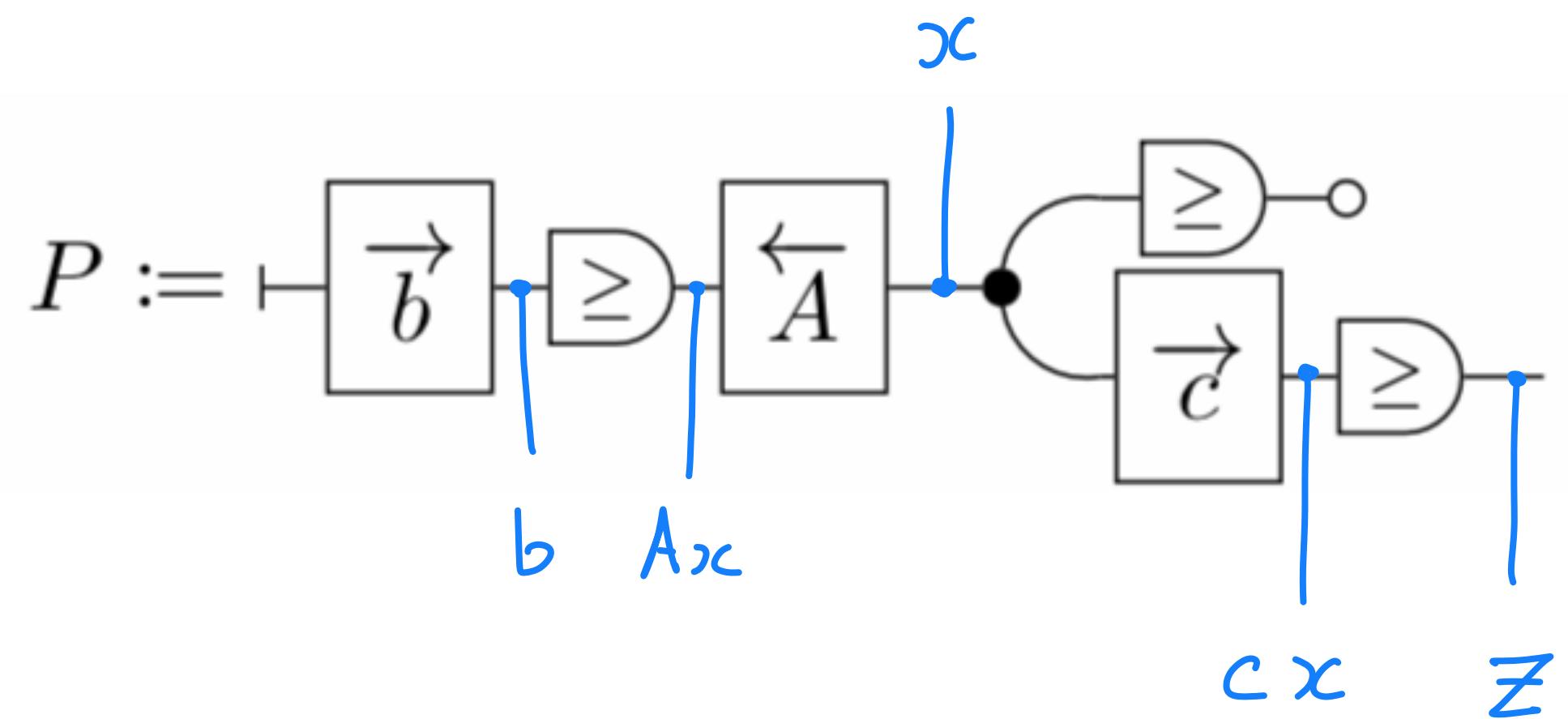
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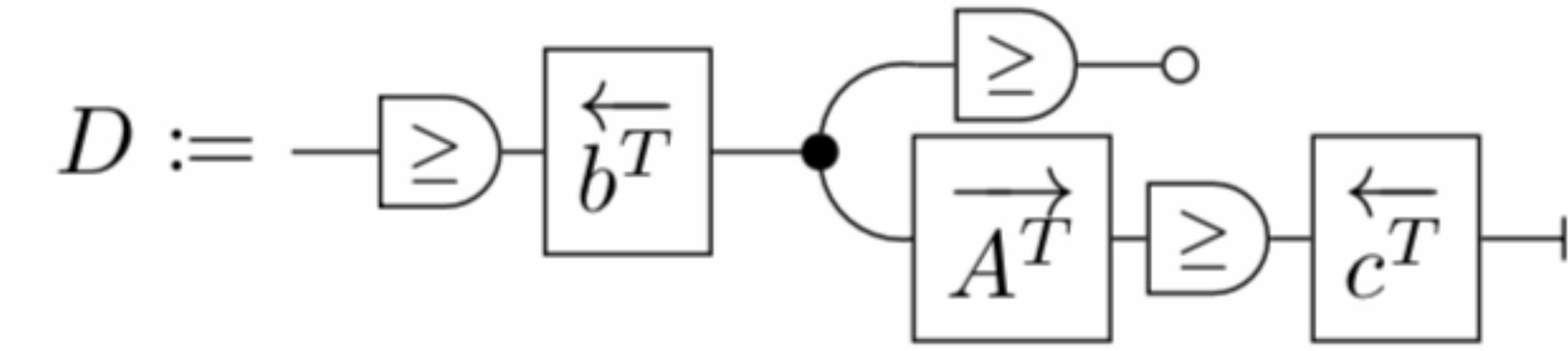
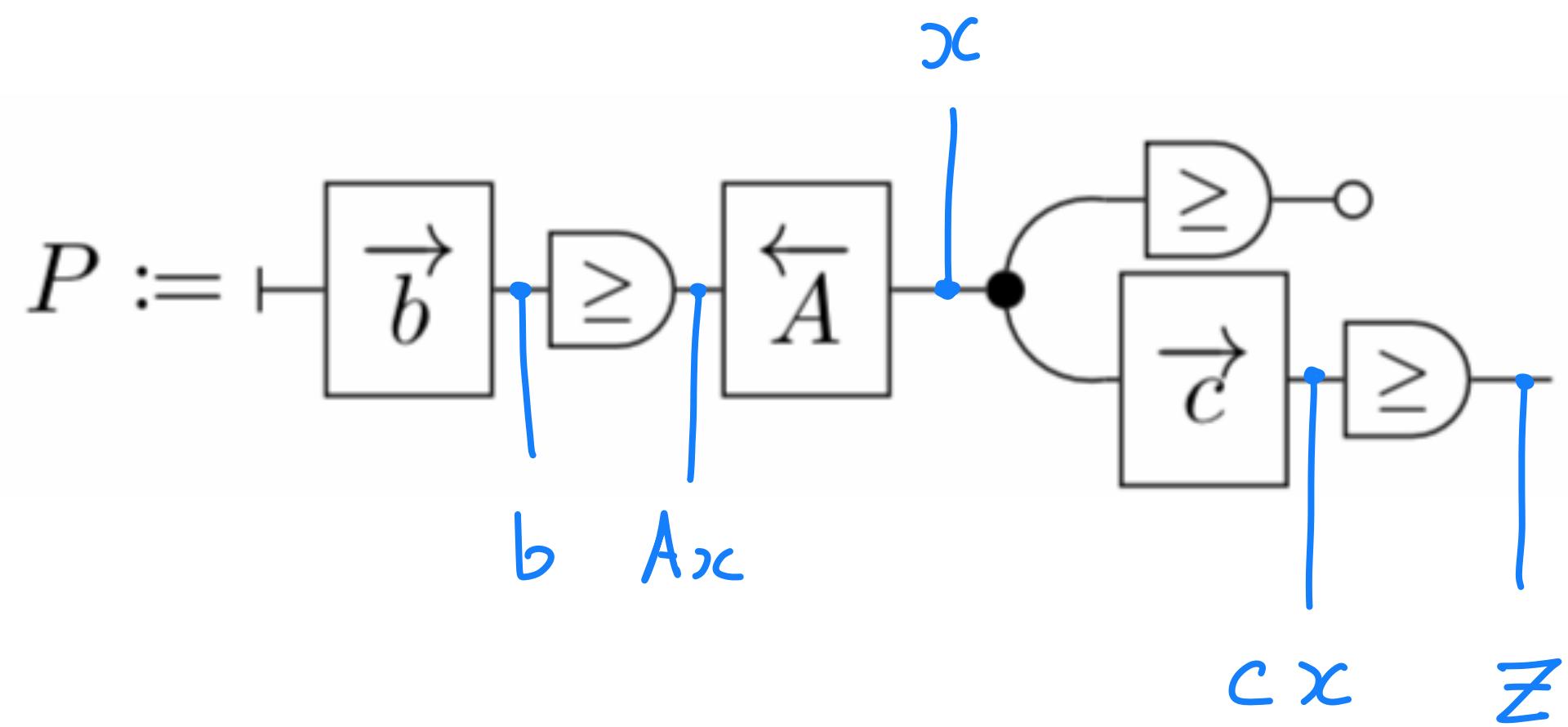


$$\llbracket P \rrbracket = \{(\bullet, z) \in \mathbb{k}^0 \times \mathbb{k}^1 \mid z \leq cx, x \geq 0, Ax \leq b\}$$

Linear programming duality

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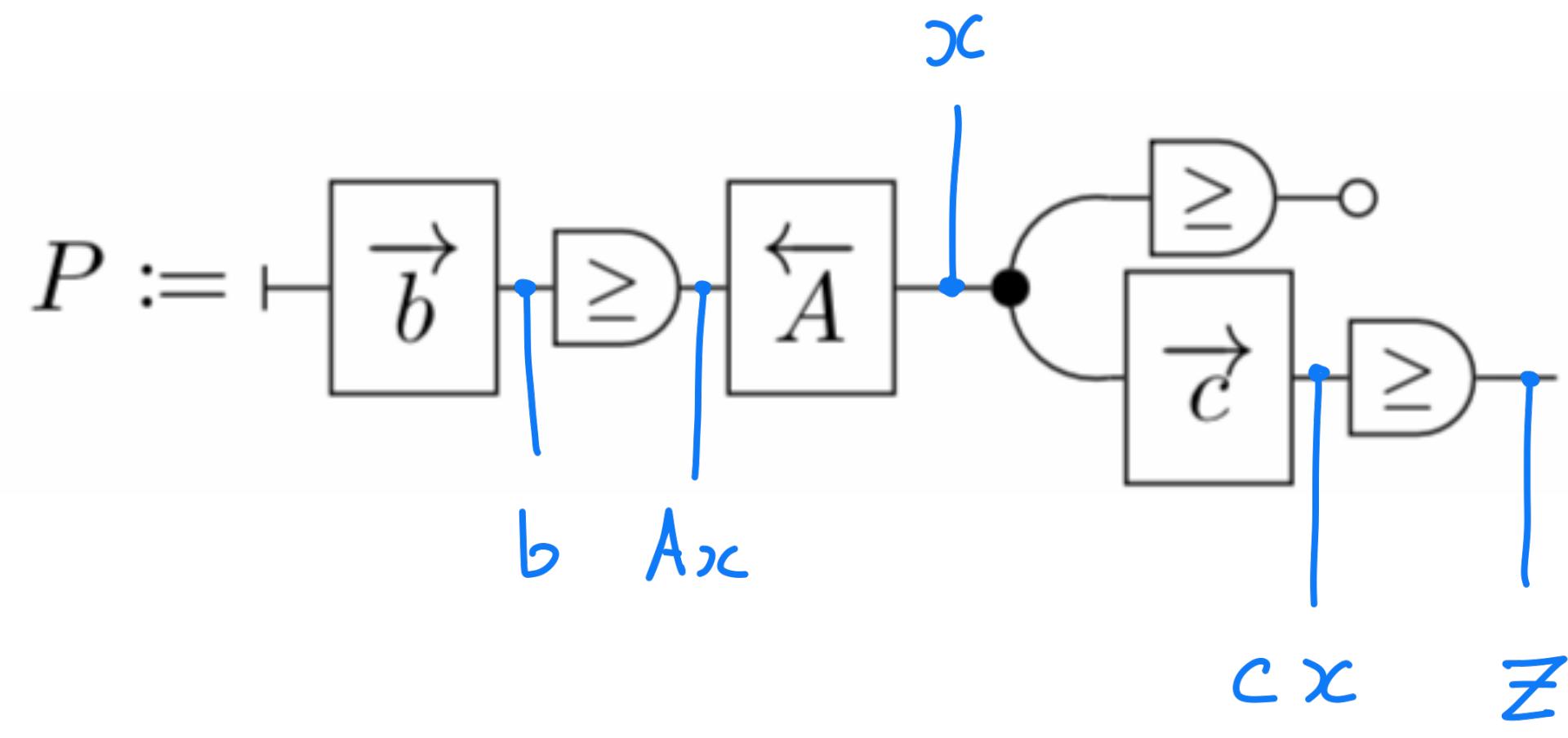
$$(D) := \min\{b^T y \mid A^T y \geq c^T \text{ and } y \geq 0\}$$



$$\llbracket P \rrbracket = \{(\bullet, z) \in \mathbb{k}^0 \times \mathbb{k}^1 \mid z \leq cx, x \geq 0, Ax \leq b\}$$

$$\llbracket D \rrbracket = \{(z, \bullet) \in \mathbb{k}^1 \times \mathbb{k}^0 \mid z \geq b^T y, y \geq 0, A^T y \geq c^T\}$$

Linear programming duality



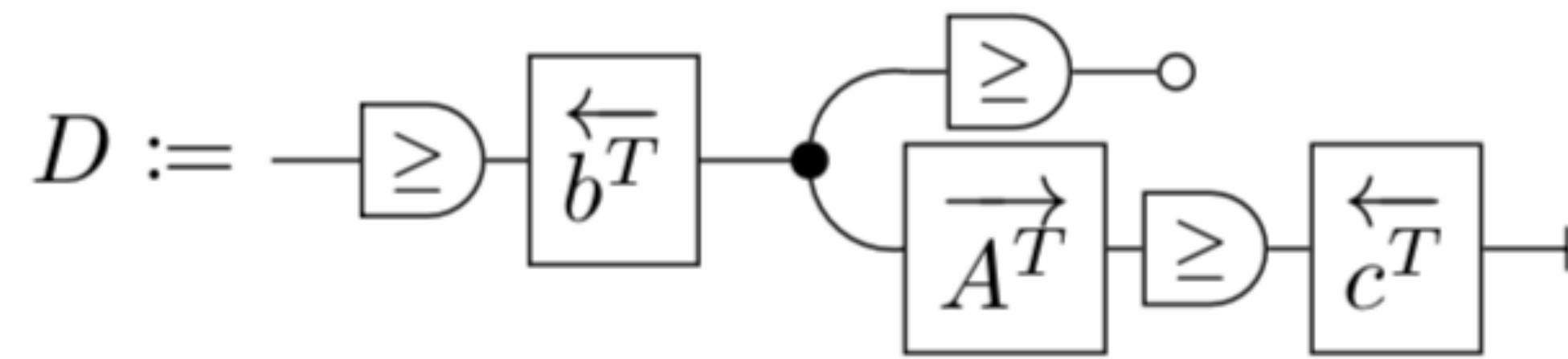
$$P \stackrel{\mathbb{PA}}{=} \vdash k \rightarrow \boxed{\geq} \rightarrow \text{ iff } k = \max\{cx \mid Ax \leq b, x \geq 0\}$$

$$P \stackrel{\mathbb{PA}}{=} \bullet \rightarrow \text{ iff } (P) \text{ is unbounded}$$

$$\llbracket P \rrbracket = \{(\bullet, z) \in k^0 \times k^1 \mid z \leq cx, x \geq 0, Ax \leq b\}$$

$$P \stackrel{\mathbb{PA}}{=} \vdash \circ \rightarrow \bullet \rightarrow \text{ iff } (P) \text{ is unfeasible}$$

Linear programming duality



$$D \stackrel{\mathbb{PA}}{=} \text{---} \begin{array}{c} \geq \\ \square \end{array} \text{---} k \text{---} \text{ iff } k = \min\{b^T y \mid A^T y \leq c^T, y \geq 0\}$$

$$D \stackrel{\mathbb{PA}}{=} \bullet \text{---} \text{ iff } (D) \text{ is unbounded}$$

$$\llbracket D \rrbracket = \{(z, \bullet) \in \mathbf{k}^1 \times \mathbf{k}^0 \mid z \geq b^T y, y \geq 0, A^T y \geq c^T\}$$

$$D \stackrel{\mathbb{PA}}{=} \bullet \text{---} \circ \text{---} \text{ iff } (D) \text{ is unfeasible}$$

Linear programming duality

(P)	(D)	bounded	unbounded	unfeasible
bounded		✓		
unbounded				✓
unfeasible			✓	✓

Linear programming duality

(P)	(D)	bounded	unbounded	unfeasible
bounded		✓		
unbounded				✓
unfeasible			✓	✓

► **Theorem 21** (Duality). *The following hold:*

1. For all $k \in \mathbb{R}$, $P \stackrel{\text{PA}}{=} \vdash \Box[k] \geq \dashv$ if and only if $D \stackrel{\text{PA}}{=} \dashv \geq \Box[k] \vdash$
2. If $P \stackrel{\text{PA}}{=} \bullet$, then $D \stackrel{\text{PA}}{=} \dashv \bullet \dashv$
3. If $D \stackrel{\text{PA}}{=} \dashv \bullet$, then $P \stackrel{\text{PA}}{=} \vdash \circ \bullet$

Questions?