Weakly Markov categories and weakly affine monads

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Introduction

Proposition

A monoid (M, m, e) in **Set** is a group if and only if the associativity square

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times \mathrm{id}} & M \times M \\ \downarrow{\scriptstyle \mathrm{id} \times m} & & \downarrow{\scriptstyle m} \\ M \times M & \xrightarrow{\quad m \quad} & M \end{array}$$

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is a pullback.

Theorem

Given a monoidal monad (T, μ, η, m) on a cartesian monoidal category, the monoid $T1$ is a group if and only if each associativity square

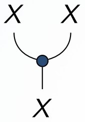
$$\begin{array}{ccc} TX \times TY \times TZ & \xrightarrow{m \times \mathrm{id}} & T(X \times Y) \times TZ \\ \downarrow{\scriptstyle \mathrm{id} \times m} & & \downarrow{\scriptstyle m} \\ TX \times T(Y \times Z) & \xrightarrow{m} & T(X \times Y \times Z) \end{array}$$

is a pullback.

GS-monoidal categories

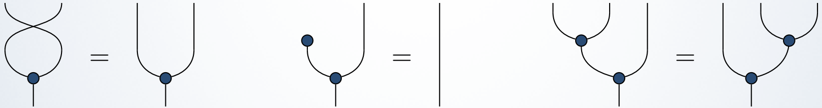
Definition

A **garbage-share (GS) monoidal category**, a.k.a. **copy-discard (CD) category**, is a SMC where each object X is equipped with maps

copy $=$ 

del $=$ 

satisfying the commutative comonoid equations



and compatible with the monoidal structure.

GS-monoidal categories

Example

The category **Set** of sets and functions has the following copy and discard maps:

$$\begin{aligned} X &\xrightarrow{\text{copy}} X \times X \\ x &\longmapsto (x, x) \end{aligned} \qquad\qquad \begin{aligned} X &\xrightarrow{\text{del}} 1 \\ x &\longmapsto \bullet \end{aligned}$$

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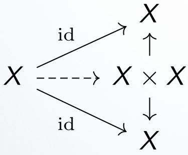
Example

More generally, in any cartesian monoidal category each object admits a unique commutative comonoid structure.



$$X \xrightarrow{\;!\;} 1$$

GS-monoidal categories

Example

The category **Rel** has

- As objects, sets;
- As morphisms, binary relations $r : X \times Y \to \{0, 1\}$;

$$s \circ r\,(x, z) = \bigvee_y r(x, y)\, s(y, z)$$

GS-monoidal categories

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Example

The category **FinStoch** has

- As objects, finite sets (or natural numbers);
- As morphisms, stochastic matrices $p : X \to Y$ of entries $p(y|x)$;

$$q \circ p\,(z|x) = \sum_y q(z|y)\, p(y|x)$$

GS-monoidal categories

Proposition

Let T be a commutative (i.e. monoidal) monad on a cartesian monoidal category $\mathbf{D}$. Then its Kleisli category $\mathbf{Kl}_T$ is canonically a gs-monoidal category with the copy and discard structure induced by that of $\mathbf{D}$.

Examples

- **Rel** is the Kleisli category of the power set monad on **Set**.

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Examples

- **Rel** is the Kleisli category of the power set monad on **Set**.
- Denote by MX the set of *finitely supported measures on* X. They form a monad which we call the **measure monad** on **Set**.
- The subset $DX \subseteq MX$ of *probability measures* also gives a monad, the **distribution monad**. Its Kleisli category admits **FinStoch** as a full subcategory.

The monoid of effects

Definition

In a gs-monoidal category we call a **state** a morphism $p : I \to X$, and **effect** a morphism $a : X \to I$.

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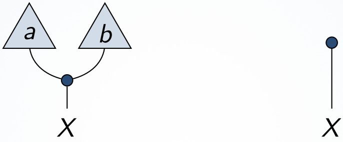
Examples

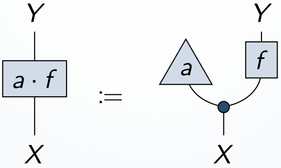
- In **Rel**, both states and effects are subsets of X.
- In $\mathbf{Kl}_M$ (and $\mathbf{Kl}_D$),
  - States are finitely supported (probability) measures on X;
  - Effects of $\mathbf{Kl}_M$ are functions $X \to [0, \infty)$;
  - For $\mathbf{Kl}_D$, the only effects are the discard maps.

The monoid of effects

Effects on a given object X form naturally a commutative monoid,



acting on morphisms $X \to Y$ via

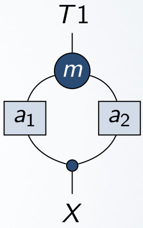


The monoid of effects

When the gs-monoidal category comes from a monad, the monoid of effects comes from the canonical monoid structure of $T1$:

$$T1 \times T1 \xrightarrow{m} T(1 \times 1) \cong T1$$

$$1 \xrightarrow{\quad\quad\eta\quad\quad} T1$$



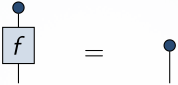
Example

For $\mathbf{Kl}_M$, $T1$ is the monoid $[0, \infty)$ with multiplication.
It acts on measures and kernels via the usual (pointwise) scalar multiplication.

Markov categories

Definition

A morphism f in a gs-monoidal category is called **full**, **discardable**, or **normalized** if and only if



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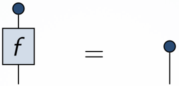

Examples

- In **Rel**, a relation $r : X \to Y$ is full if and only if each $x \in X$ is related to at least one $y \in Y$. (This is the usual definition of full relation.)
- In $\mathbf{Kl}_M$, a matrix $m : X \to Y$ is full if and only if it is stochastic, i.e. each column is normalized:
$$\sum_{y \in Y} m(y|x) = 1.$$

Markov categories

Definition

A gs-monoidal category is called **Markov** if any of the following equivalent conditions hold:

- It is affine monoidal (i.e. the monoidal unit I is terminal);
- The only effects are the discard maps;
- The discard maps form a natural transformation $\mathrm{id} \Rightarrow \Delta_I$;
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$\mathbf{Kl}_D$ is a Markov category, and so is its subcategory **FinStoch**.

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Proposition

Let T be a commutative monad on a cartesian monoidal category. The GS monoidal category $\mathbf{Kl}_T$ is Markov if and only if $T1 \cong 1$.

Weakly Markov categories

Definition

A gs-monoidal category $\mathbf{C}$ is called **weakly Markov (WM)** if for every object X, the monoid of effects $\mathbf{C}(X, I)$ is a group.

Definition

A commutative monad T on a cartesian monoidal category is called **weakly affine** if the monoid $T1$ is a group.

Weakly Markov categories

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Definition

A commutative monad T on a cartesian monoidal category is called **weakly affine** if the monoid $T1$ is a group.

Proposition

A commutative monad on a cartesian monoidal category is weakly affine if and only if its Kleisli category is weakly Markov.

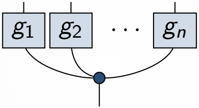
Example

Let $M^*X \subseteq MX$ be the set of *nonzero* measures on X.
This forms a weakly affine submonad $M^* \subseteq M$.

Conditional independence in WMCs

Definition

A morphism $f : A \to X_1 \otimes \cdots \otimes X_n$ in a gs-monoidal category is said to exhibit **conditional independence of the X_i given** A if and only if it can be expressed as a product of the following form



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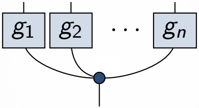

Proposition

Let $f : A \to X_1 \otimes \cdots \otimes X_n$ be a morphism in a *weakly Markov* category. Then f exhibits conditional independence of the X_i given A if and only if it is *in the same orbit* as the product of all its marginals.

Conditional independence in WMCs

Lemma (localised independence property)

Whenever a morphism $f : A \to X \otimes Y \otimes Z$ in a WM category exhibits conditional independence of $X \otimes Y$ (jointly) and Z, as well as conditional independence of X and $Y \otimes Z$, then it exhibits conditional independence of X, Y, and Z (all given A).

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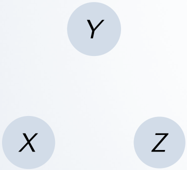

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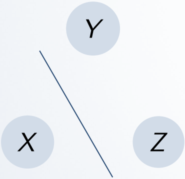

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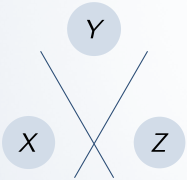

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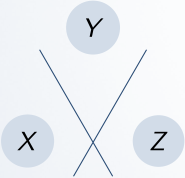
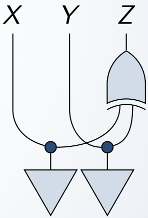


but compare:



Main statement

Theorem

Let T be a commutative monad on $\mathbf{D}$ on a cartesian monoidal category. Then the following conditions are equivalent

1. T is weakly affine;
2. the Kleisli category $\mathbf{Kl}_T$ is weakly Markov;
3. for all objects X, Y, and Z, the following associativity diagram is a pullback:

$$\begin{array}{ccc} TX \times TY \times TZ & \xrightarrow{m \times \mathrm{id}} & T(X \times Y) \times TZ \\ \Big\downarrow{\scriptstyle \mathrm{id} \times m} & & \Big\downarrow{\scriptstyle m} \\ TX \times T(Y \times Z) & \xrightarrow{m} & T(X \times Y \times Z) \end{array}$$

Conclusion

- Intermediate theory:

$$\begin{array}{ccccc}
\text{Markov cats} & \hookrightarrow & \textbf{WM cats} & \hookrightarrow & \text{gs-m. cats} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\text{affine c. monads} & \hookrightarrow & \textbf{weakly affine c. monads} & \hookrightarrow & \text{c. monads}
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- Extension of conditional independence to WM case (useful e.g. for probabilistic programming)
- Future work: extension of more probabilistic concepts (positivity, causality, etc.) as well as interaction with nontrivial effects.

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