Structural Operational Semantics for Heterogeneously Typed Coalgebras

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Background and Research Question

- Joint Research Program at UiB and HVL, Bergen
  - Coordination of concurrently interacting components in modular software architectures
  - Purpose: Correctness checks w.r.t. global consistency rules and ...
  - ...later: (Semi-)automatic repair (proposals)
- State of the project
  - Component alignment generates amalgamated graph transformation system¹
  - But carried out by ad-hoc implementations depending on the involved behaviours

What is a general coordination framework for capturing the dynamics of interacting, arbitrarily typed behavioural components?

Traffic Control: (Local) Components

- Traffic Controller: Requests traffic light changes, receives sensor signals
- Traffic Lights: Change state passively
- Buses: Probability distribution in simulation scenario
- Tram: Timed Processes
Traffic Control: Compound System

Coordination Language, Generation of Compound Behaviour
Coalgebras for General Dynamical Systems

Heterogeneous behaviour: Different $\mathcal{SET}$-endofunctors

- Traffic Controller: $\mathcal{B}_1 = (1 + O \times \_)^E$
- Traffic Lights: $\mathcal{B}_2 = (1 + \_)^I$
- Buses: $\mathcal{B}_3 = (1 + \mathcal{D}_\omega(\_))$
- Tram: $\mathcal{B}_4 = (\_)^T$
- Compound System: $\mathcal{B} = ?$
Distributive Laws

Generating Global Behaviour

- **Green light, tram approaching** ⇒ **Red light, tram passing**
- I.e. inference rule

\[
\begin{align*}
\text{premises} & \\
\text{conclusion} & 
\end{align*}
\]
Capturing Compound Behaviour: SOS-Laws

For $B$ an intended behavioural interpretation, $\Sigma$ process term syntax, e.g.

\[
\begin{align*}
\frac{a.x}{a \cdot x} & \xrightarrow{a} x \\
\frac{x \xrightarrow{a} x'} y \xrightarrow{b} y' & \xrightarrow{x \mid y} x' \mid y \\
\frac{x \xrightarrow{a} x'} y \xrightarrow{b} y' & \xrightarrow{x \mid y} x' \mid y'
\end{align*}
\]

In the above example:

- $\Sigma(X) = 1 + \bigsqcup_{a \in A} X + X^2$
- $B(X) = \wp_{fin}(X)^A$, $\mathcal{H}(X) := X \times B(X)$

Encoding of rules (shown only for $\mid\mid$):

$\rho_X : \left\{(X \times \wp_{fin}(X)^A)^2 \rightarrow \wp_{fin}(1 + \bigsqcup_{a \in A} X + X^2)^A \right\}$

$(x, \beta_1, y, \beta_2) \mapsto \lambda c. \{ (x', y) \mid x' \in \beta_1(c) \} \cup \{ (x, y') \mid y' \in \beta_2(c) \}$

Similarly for the other summands of $\Sigma$: $\rho : \Sigma \mathcal{H} \Rightarrow B \Sigma$.

**Σ-Terms of Behaviour ⇒ Behaviour of Σ-Terms**
Capturing Compound Behaviour: Distributive Laws

**Definition (Distributive Law)**

Let $\mathcal{C}$ be a category with products, $\Sigma, \mathcal{B} : \mathcal{C} \to \mathcal{C}$ and $\mathcal{H}(X) := X \times \mathcal{B}(X)$. A Distributive Law of $\Sigma$ over $\mathcal{H}$ is a natural transformation $\lambda : \Sigma \mathcal{H} \Rightarrow \mathcal{H} \Sigma$, which is compatible with $(\pi_1, X : X \times \mathcal{B}(X) \to X)_{X \in \Sigma \mathcal{E} \mathcal{T}}$, i.e. $\pi_1, \Sigma \circ \lambda = \Sigma \pi_1$.

**Theorem**

Natural Transformations $\rho : \Sigma \mathcal{H} \Rightarrow \mathcal{B} \Sigma \stackrel{1:1}{\leftrightarrow}$ Distributive Laws over $\mathcal{H}$.

**Remark:**

**Theorem (Turi, Plotkin, 1997)**

For the above example, image-finite GSOS rule sets are in $1:1$-correspondence to natural transformations $\Sigma \mathcal{H} \Rightarrow \mathcal{B} \Sigma^*$. 
Compositionality and Bialgebras

- \((x_i \sim x'_i)_{i \in \{1,...,n\}} \Rightarrow op(x_1, \ldots, x_n) \sim op(x'_1, \ldots, x'_n)\)
- In our context: Is observational equivalence preserved after the construction of the compound system?

For \(\lambda : \Sigma \mathcal{H} \Rightarrow \mathcal{H} \Sigma : \mathbb{C} \to \mathbb{C}\) and \(X \xrightarrow{\alpha} \mathcal{H}(X)\) there is the \(\lambda\)-induced coalgebra

\[
\Sigma(X) \xrightarrow{\Sigma \alpha} \Sigma \mathcal{H}(X) \xrightarrow{\lambda_X} \mathcal{H} \Sigma(X)
\]

Furthermore, there are two important diagrams:

\[
\begin{align*}
\Sigma(A) \xrightarrow{init} & A \\
\downarrow \Sigma h_\lambda & \downarrow h_\lambda \\
\Sigma \mathcal{H}(A) \xrightarrow{\lambda_A} & \mathcal{H} \Sigma(A) \xrightarrow{\mathcal{H} init} \mathcal{H}(A)
\end{align*}
\]

\[
\begin{align*}
\Sigma(A) \xrightarrow{init} & A \xrightarrow{h_\lambda} \mathcal{H}(A) \\
\downarrow \Sigma f & \downarrow f \\
\Sigma(Z) \xrightarrow{g_\lambda} & Z \xrightarrow{\zeta} \mathcal{H}(Z)
\end{align*}
\]

Theorem (Klin, 2011)

*Observational Equivalence is a Congruence.*
Challenges, Refined Research Question

Facts:

- Inference rules inductively determine provable behaviour of all $\Sigma$-terms from atomic transitions.
- Bialgebraic Theory provides general proof for compositionality.
- Transition rules act on a single state space.

Challenges:

- We want to avoid recurrent term generation and only determine behaviour of the generated compound system.
- With an adjusted approach for interacting individual components, can we still guarantee compositionality?
- In heterogeneous specifications, state spaces must be kept separate!

How can we apply the bialgebraic theory to formally understand interacting, heterogeneously typed behavioural components?
Many-Sortedness and Holistic Approach

Non-recurrent "term"-generation via different sorts:

- Each "attempt" to let certain components interact, is represented by an operation $\text{op} : s_1 \cdots s_n \rightarrow s_{n+1}$
- In an $n + 1$-sorted algebra, each carrier of sort $s \in \{s_1, \ldots, s_n\}$ represents the state space of a local component, and ... 
- ... the carrier for sort $n + 1$ represents the state space of the compound system.

$\rightarrow$ Algebras simultaneously describe the local and global state spaces.
Interaction Law Instead of Distributive Laws

Let \((S_i, \alpha_i) \in \mathcal{B}_i\text{-}\text{Coalg})_{i \in \{1, \ldots, n\}}\) be the local components and \(\mathcal{B}\) be the behavioural specification of the compound system.

*Keeping state spaces separate* in a rule-induced coalgebra \((n = 2)\):

\[
X_1 \times X_2 \xrightarrow{\langle id, \alpha_1 \rangle \times \langle id, \alpha_2 \rangle} X_1 \times \mathcal{B}_1(X_1) \times X_2 \times \mathcal{B}_2(X_2) \xrightarrow{\rho_{x_1, x_2}} \mathcal{B}(X_1 \times X_2)
\]

In general \(X_1 \times X_2\) is replaced by an arbitrary set (\(\mathbb{C}\text{-object}\)) constructed out of \(n\) input sets:

\[
\Sigma : \text{SET}^n \to \text{SET}
\]

**Definition (Interaction Law)**

An *interaction law* is a natural transformation

\[
\rho : \Sigma(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \Rightarrow \mathcal{B}\Sigma : \text{SET}^n \to \text{SET}.
\]

But this is apparently no longer an interplay between endofunctorial syntax and a single behaviour!
Example: Heterogeneous SOS

\[
\begin{pmatrix}
  x_1 \xrightarrow{e/o} x'_1 & x_2 \xrightarrow{i} x'_2 \\
  \text{op}(x_1, x_2) \xrightarrow{\varphi(o,i)} \text{op}(x'_1, x'_2)
\end{pmatrix}
\]

\( o \in O, i \in I \)

As interaction law:

\[
\rho_{X_1, X_2} : X_1 \times (1 + O \times X_1)^E \times X_2 \times (1 + X_2)^I \rightarrow \varphi_{fin}(X_1 \times X_2)^{Act}
\]

\[
(x_1, \beta_1, x_2, \beta_2) \mapsto \\
\left\{ (\tau, (x'_1, x'_2)) \mid (o, x'_1) \in \beta_1(E), x'_2 = \beta_2(i) \right\} \cup \\
\left\{ (o, (x'_1, x_2)) \mid (o, x'_1) \in \beta_1(E), o \text{ unsynchr.} \right\} \cup \\
\left\{ (i, (x_1, x'_2)) \mid x'_2 = \beta_2(i), i \text{ unsynchr.} \right\}
\]
The Main Result

Uses classical results for endofunctors despite state space separation!

**Theorem**

Let \( (S_i \xrightarrow{\alpha_i} B_i(S_i) \in B_i\text{-Coalg})_{i \in \{1, \ldots, n\}} \) and \( B \) be the behavioural specification of the compound system. Let them all admit final objects. Let \( \Sigma : \text{SET}^n \rightarrow \text{SET} \). Compositionality holds, if the computation of the compound system can be described by an interaction law \( \rho : \Sigma(H_1 \times \cdots \times H_n) \Rightarrow B\Sigma \).

Proof idea:

- Holistic behaviour: \( \vec{B} := \Pi_{1 \leq i \leq n} B_i \times B : \text{SET}^{n+1} \rightarrow \text{SET}^{n+1} =: \mathbb{C} \)
- Lifting local behaviour to global behaviour: \( \vec{\Sigma} : \mathbb{C} \rightarrow \mathbb{C} \) def. by \( \vec{\Sigma}(X_1, \ldots, X_n, X_{n+1}) := (S_1, \ldots, S_n, \Sigma(X_1, \ldots, X_n)) \)
- Fixing locals: \( \vec{\rho} := (\alpha_1, \ldots, \alpha_n, \rho) : \vec{\Sigma} \vec{H} \Rightarrow \vec{B} \vec{\Sigma} : \mathbb{C} \rightarrow \mathbb{C} \) Option
- Thus setting of \( 10 \), which yields distributive law \( \vec{\lambda} : \vec{\Sigma} \vec{H} \Rightarrow \vec{H} \vec{\Sigma} \) and with classical results the desired result. \( \Box \)
Related Work

- Practical Approaches
  - Co-simulation
  - Coordination Languages

- Coalgebraic Abstraction of SOS Framework
  - Klin’s Survey
  - Categorically in B. Jacobs’ book

- Heterogeneity
  - M. Kick, J. Power, A. Simpson *Coalgebraic semantics for timed processes.*
  - …

- (Co-)Institutions

See references in the paper
Résumée and Future Work

Holistic many-sorted formal approach for concurrently interacting heterogeneously typed coalgebras. Evaluation of the approach by proving compositionality.

Future Work:

- Implementation viewpoint: Currently very cumbersome
- Intermediate interaction: Sort inflation
- Extensions: Behaviour: Name passing, Syntax: Equational specifications
- Adequate (co-)institutional methods
- Aspects of Temporal Constraints
Appendix

Optional Slide to Explain Action of $\vec{\rho}$

\[
\begin{align*}
(S_i \xrightarrow{\alpha_i} B(S_i))_{i \in \{1, \ldots, n\}}, \rho : \Sigma(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) & \Rightarrow B\Sigma. \\
\vec{B} & := \Pi_{1 \leq i \leq n} B_i \times B : \text{SET}^{n+1} \rightarrow \text{SET}^{n+1} \\
\vec{H} & := \Pi_{1 \leq i \leq n} H_i \times H : \text{SET}^{n+1} \rightarrow \text{SET}^{n+1} \\
\vec{\Sigma}(X_1, \ldots, X_n, X_{n+1}) & := (S_1, \ldots, S_n, \Sigma(X_1, \ldots, X_n))
\end{align*}
\]

Then it is defineable as

\[
\vec{\rho}_{X_1, \ldots, X_{n+1}} : \vec{\Sigma}\vec{H}(X_1, \ldots, X_{n+1}) \rightarrow \vec{B}\vec{\Sigma}(X_1, \ldots, X_{n+1})
\]

because

\[
\begin{align*}
\vec{\rho}_{X_1, \ldots, X_{n+1}} : S_1 & \times \cdots \times S_n \times \Sigma(\mathcal{H}_1(X_1), \ldots, \mathcal{H}_n(X_n)) \\
\downarrow \alpha_1 & \cdots \downarrow \alpha_n \\
\rightarrow B_1(S_1) & \times \cdots \times B_n(S_n) \times B\Sigma(X_1, \ldots, X_n)
\end{align*}
\]

i.e. $\vec{\rho} := (\alpha_1, \ldots, \alpha_n, \rho) : \vec{\Sigma}\vec{H} \Rightarrow \vec{B}\vec{\Sigma}$.
Optional: Adapted Notion of Congruence

Let $A_1, ..., A_n, A$ be sets and

$$f : \Sigma(A_1, ..., A_n) \to A$$

be a map. A family of binary relations

$$(R_i \subseteq A_i \times A_i)_{i \in \{1, ..., n\}}, R \subseteq A \times A$$

is said to be $f$-compatible, if there is a map $r$, such that the following diagram commutes:

$$\begin{array}{ccc}
\Sigma(A_1, ..., A_n) & \xleftarrow{\Sigma(\pi^1_1, ..., \pi^n_1)} & \Sigma(R_1, ..., R_n) & \xrightarrow{\Sigma(\pi^1_2, ..., \pi^n_2)} & \Sigma(A_1, ..., A_n) \\
\downarrow f & & \downarrow r & & \downarrow f \\
A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & A
\end{array}$$