Generic Trace Semantics and Graded Monads

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CALCO 2015

Introduction

- Coalgebra does bisimilarity
- Traces need algebra: impose additional equational laws
  - See [Kurz/Milius/Pattinson/Schröder 2015] for trace semantics via monads
- Here: refine this to use graded monads
  - control over trace length
- Generalize all previous approaches
- Introduce graded algebras
  - obtain generic trace logic

Trace Semantics and Algebraic Laws

LTS = Coalgebras for functor $GX = \mathcal{P}(\Sigma \times X)$.
Algebraically: Shallow theory

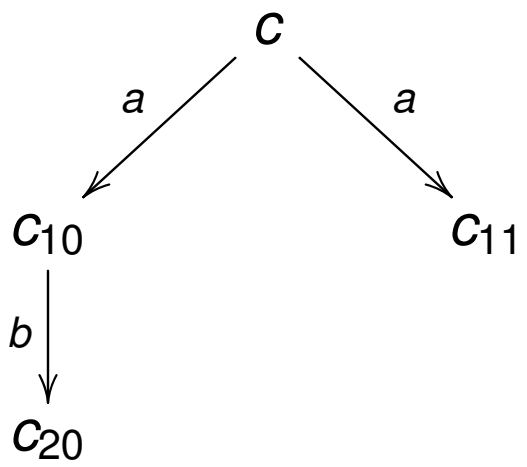
- Operations $\sum a_i(\cdot)_i$
- Shallow laws (e.g. $a(x) + b(y) = b(y) + a(x)$)

Trace semantics:

- Split operations : $\sum$, a
- JSL laws for $\sum$
- Distributivity:

$$a(\sum x_i) = \sum a(x_i),$$

Trace Semantics, Incrementally



Pretrace = Pair (u, d), u word, d state.

	Pretraces	Traces
Stage 0 :	$\{(\varepsilon, c)\}$	$\{\varepsilon\}$
Stage 1 :	$\{(a, c_{10}), (a, c_{11})\}$	$\{a\}$
Stage 2 :	$\{(ab, c_{20}))\}$	$\{ab\}$
Stage 3 :	$\emptyset$	$\emptyset$

Graded Monads

(Smirnov 2008)

- Functors $M_n : \mathbf{C} \to \mathbf{C}$
  - Think: $M_n X =$ Terms of uniform depth n over X
- Unit $\eta : id \to M_0$
- Multiplication $\mu^{nk} : M_n M_k \to M_{n+k}$
- Correspond to graded theories
  - operators with assigned depths (e.g. 0, 1)
  - Uniform-depth equations

Here: depth $=$ trace length.

Traces Semantics Via Graded Monads

Trace semantics of G =

- Graded monad (M_n)
- Natural transformation $\alpha : G \to M_1$

For coalgebra $\gamma : X \to GX$ α-pretrace maps

$$\gamma^{(0)} = \eta : X \to M_0 X$$

$$\gamma^{(n+1)} = X \xrightarrow{\alpha\gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n X \xrightarrow{\mu_X^{1n}} M_{n+1} X$$

and α-trace maps

$$X \xrightarrow{\gamma^{(n)}} M_n X \xrightarrow{M_n !} M_n 1$$

Trace Semantics: Examples

- Finite-depth bisimilarity: $M_n = G^n$;
  trace maps $X \to G^n 1 =$ cone into final sequence.
- Trace semantics of LTS: $M_n X = \mathcal{P}(\Sigma^n \times X)$
- Mazurkiewicz traces
- Probabilistic traces:
  - Generative probabilistic transition systems: $G = \mathcal{D}(\Sigma \times X)$
  - $M_n = \mathcal{D}(\Sigma^n \times X)$ from
    $$a(\sum p_i x_i) = \sum p_i a(x_i)$$

Example: Trace Semantics, Kleisli Style

(Hasuo/Jacobs/Sokolova 2007)

E.g. LTS with explicit termination (NDA) are TF-coalgebras for

$$T = \mathcal{P} \qquad F = 1 + \Sigma \times \_$$

Kleisli law ($\cong$ lifting $\bar{F}$ of F to Kleisli category)

$$\lambda_X : FT \to TF$$

+ some assumptions $\to$ language semantics via final map

$$X \to T\nu\bar{F} \cong T\mu F = \mathcal{P}(\Sigma^*)$$

(becomes trivial for plain LTS, i.e. $F = \Sigma \times \_$)

- TF^n is graded monad
- α-trace maps $X \to TF^n 1 \cong \mathcal{P}(\sum_{i \leq n} \Sigma^i)$: traces and accepted words
- Language semantics by canonical forgetting

Example: Trace Semantics, Eilenberg-Moore Style

(Bonsangue/Milius/Silva 2013)

E.g. LTS with explicit termination (NDA) are FT-coalgebras for

$$T = \mathcal{P} \qquad F = 2 \times \_^{\Sigma}$$

EM-law ($\cong$ lifting $\hat{F}$ of F to EM category)

$$\rho : TF \to FT$$

Language semantics: Final Map

$$X \xrightarrow{\eta} TX \to U\nu\hat{F} \cong \nu F = \mathcal{P}(\Sigma^*)$$

(becomes trivial for plain LTS, i.e. $F = X^{\Sigma}$)

- $F^n T$ is graded monad
- α-trace maps $X \to F^n T1 = F^n \mathcal{P}1$: traces and accepted words
- Language semantics by canonical forgetting

Graded Algebras

M_n-algebra for $n < \omega$ (similarly for $n = \omega$):

- Objects A_k, $k \leq n$
- Maps

$$a^{mk} : M_m A_k \to A_{m+k} \qquad (m + k \leq n)$$

Trace Properties

$(M_k X)_{k \leq n}$ is the free M_n-Algebra over X.

- $\to$ M_n-Algebras + one truth value $\tau : 1 \to A_n$ are trace properties:

$$X \xrightarrow{M_n!\gamma(n)} M_n 1 \xrightarrow{\tau_n^\#} A_n$$

- Compositional syntax?
  - That's what trace logics are about:
    $\langle a \rangle \top \wedge \langle b \rangle \top$ is trace-invariant, $\langle a \rangle (\langle a \rangle \top \wedge \langle b \rangle \top)$ is not.

Depth-1 Graded Monads

- Depth-1 graded theory:
  - all operations and equations have depth ≤ 1
- Depth-1 graded monad:
  - μ^{nk} and $M_0\mu^{1n}$ are pointwise epi
  - μ^{1n} is a (reflexive) coequalizer

$$\begin{tikzcd} M_1 M_0 M_n \arrow[r, shift left, "M_1\mu^{0n}"] \arrow[r, shift right, "\mu^{10}M_n"'] & M_1 M_n \arrow[r, "\mu^{1n}"] & M_{1+n}\,. \end{tikzcd}$$

In depth 1, M_n-algebras are compatible chains of M_1-algebras

Generic Trace Logics

Formulas:

- Depth 0: Constants c
- Depth $n+1$: $L(\phi_1, \ldots, \phi_n)$, L modal operator, ϕ_i depth n.

Semantics:

- M_0-algebra Ω of truth values
- Truth values $[\![c]\!] : 1 \to \Omega$
- M_1-algebras $[\![L]\!] : M_1(\Omega^n) \to \Omega$.
- $[\![\phi]\!]$ is trace property (M_n-algebra, truth value)

Trace logics: Examples

- LTS ($G = \mathcal{P}(\Sigma \times X)$):
  - Truth value object: 2 with the usual $\mathcal{P}$-algebra structure (complete JSL)
  - Unary operators: e.g.

$$[\![\langle a \rangle]\!] : \mathcal{P}(\Sigma \times 2) \to 2, \quad S \mapsto \begin{cases} \top & (a, \top) \in S \\ \bot & \text{otherwise} \end{cases}$$

  - Generally: $[\![L]\!]$ interprets $a \in \Sigma$ as join-cont. map $2^n \to 2$
    $\to$ add disjunction
- Finite-depth bisimilarity ($M_n = G^n$):
  - Set 2 of truth values
  - k-ary operators: maps $[\![L]\!] : G(2^k) \to 2$ are k-ary predicate liftings
    - closed under Boolean operators

    $\to$ coalgebraic modal logic

Example: Probabilistic Trace Logic

For $G = \mathcal{D}(\Sigma \times X)$:

- Set $[0,1]$ of truth values
- Unary operators: e.g.

$$\begin{aligned} [\![\langle a \rangle]\!] : \mathcal{D}(\Sigma \times [0,1]) &\to [0,1] \\ \mu &\mapsto \textstyle\sum_{p \in [0,1]} p\mu(a,p) \end{aligned}$$

- Generally: k-ary operator interprets a as convex morphism $[0,1]^k \to [0,1]$, i.e. affine map
- Thus: add affine combinations, including fuzzy negation $1 - \phi$.
- Precisely:

$$\phi ::= \textstyle\sum_{a \in \Sigma} \langle a \rangle (c_a + \sum q_{ai} \phi_{ai})$$

where $c_a + \sum q_{ai}(\cdot)_{ai} : [0,1]^k \to [0,1]$.

Conclusions and Future Work

- Trace semantics via graded monoids
  - subsumes existing approaches to generic finite traces
- Graded algebras are formulae in trace logics
  - Cover proper probabilistic systems
- Future work:
  - Expressivity
  - Temporal extensions
  - Model checking
  - Reasoning