The Locally Finite Fixpoint

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For an endofunctor $H : \mathcal{C} \to \mathcal{C}$, coalgebras $C \xrightarrow{c} HC$ present an elegant and generic way to talk about state and equation systems of type H. The final H-coalgebra captures all *behaviours* or *solutions* since each H-coalgebra induces a unique coalgebra homomorphism into it. However, in many applications, one is interested in *finite* or *finitely generated* systems.

We present here a new semantic domain for the behaviour of finite coalgebraic systems: the *Locally Finite Fixpoint (LFF)*. This fixpoint of the given functor H is characterized by two universal properties (as an algebra and as a coalgebra for H), and we provide a coalgebraic construction of it. As concrete instances of the LFF we obtain instances of the rational fixpoint – a previously known (co)algebraic domain for finite behaviour – but also examples like context-free languages and algebraic trees over a signature that (most likely) do not arise as instances of the rational fixpoint.

1 Basic Categorical Notions and Assumptions

We work with locally finitely presentable (lfp) categories. In such a category two notions of "finite object" are present: finitely presentable (f.p.) ones and finitely generated (f.g.) ones. We restrict ourselves to working with mono-preserving finitary endofunctors. This implies the existence of a final coalgebra, but our aim is to characterize a coalgebra that captures precisely the behaviours of coalgebras with an f.g. carrier.

2 Existing Work

The *rational fixpoint* $r : \varrho H \to H\varrho H$ of a finitary $H : \mathcal{C} \to \mathcal{C}$, $\mathcal{C}$ lfp, is introduced in [1]. It is constructed as the filtered colimit of all H-coalgebras with f.p. carrier and is a fixpoint of H. Its inverse r^{-1} has an algebraic universal property: it is the *initial* iterative algebra. Moreover, r also is the *final* locally finitely presentable coalgebra, where a coalgebra is called lfp if it is the filtered colimit of coalgebras with f.p. carrier. Concrete instances of the rational fixpoint range over regular languages, regular trees for a signature, rational λ-trees modulo α-equivalence, and rational streams (or weighted languages).

However, in categories where f.g. and f.p. objects do not coincide, ϱH may not be subcoalgebra of νH. I.e. there are examples in which the rational fixpoint does not identify all equivalent behaviours.

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To solve this, we replace "f.p." in definitions and constructions for the rational fixpoint by "f.g.". The H-coalgebras with f.g. carrier form a category denoted $\mathsf{Coalg}_{\mathrm{fg}} H$. A coalgebra $c : C \to HC$ is called *locally finitely generated (lfg)*, if any f.g. subobject $f : S \rightarrowtail C$ of the carrier factors through a subcoalgebra of (C, c) with f.g. carrier. The category of lfg coalgebras is denoted by $\mathsf{Coalg}_{\mathrm{lfg}} H$. An algebra $\alpha : HA \to A$ is called fg-iterative if for

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any morphism $e : X \to HX + A$ with X f.g. there exists a unique $e^\dagger : X \to A$ such that $e^\dagger = [\alpha, \mathrm{id}_A] \cdot (He^\dagger + \mathrm{id}_A) \cdot e$. The fg-iterative algebras together with all H-algebra homomorphisms form a category. Our main result is the following.

▶ **Theorem 1.** *Let $H : \mathcal{C} \to \mathcal{C}$ be a mono-preserving finitary functor on the lfp category $\mathcal{C}$. Then for a coalgebra $\ell : L \to HL$, the following are equivalent:*

1. *(L, ℓ) is the final lfg coalgebra.*
2. $(L, \ell) \cong \mathrm{colim}(\mathsf{Coalg}_{\mathrm{fg}} H \hookrightarrow \mathsf{Coalg} H)$
3. *The morphism ℓ is invertible and ℓ^{-1} is the initial fg-iterative algebra.*

The final lfg coalgebra (L, ℓ) exists and moreover fulfills:

(a) *(L, ℓ) is a fixpoint for H, i.e. ℓ is isomorphic.*
(b) *(L, ℓ) is a subcoalgebra of νH.*

Points 1. and 3. establish two universal properties, while point 2. provides a coalgebraic construction. The fact that the LFF is always a subcoalgebra of νH means that the LFF captures precisely behavioural equivalence of all f.g. carried H-coalgebras.

4 Applications

Whenever the classes of f.p. and f.g. objects coincide in $\mathcal{C}$ the LFF is isomorphic to the rational fixpoint. So the LFF captures all the concrete examples of the rational fixpoint we mentioned in Section 2. We now describe instances of the LFF that are not known (and we believe they are unlikely) to be captured by the rational fixpoint.

4.1 The Generalized Determinization

For a finitary Set-monad T and for the endofunctor $H = B \times (-)^A$, [4] describes a way to transform a HT-coalgebra $x : X \to HTX$ into a $\hat{H}$-coalgebra $x^\sharp : TX \to HTX$, where $\hat{H}$ is a lifting of H to the Eilenberg-Moore category Set^T (assuming a lifting exists). Depending on the choice of B and T we get different notions of machines. E.g. for $T = \mathcal{P}_\omega$ and $B = \{0, 1\}$ we get the ordinary powerset construction of non-deterministic automata. One can show that the LFF of $\hat{H}$ in Set^T is carried by the union of behaviours of finite determinized HT-coalgebras.

- For T denoting the non-deterministic stack-monad and B consisting of functions mapping the topmost stack elements to truth values, HT-coalgebras are precisely the stack-machines [3]. The LFF of $\hat{H}$ yields the context-free languages over A.
- For $T = \mathcal{P}_\omega(((-) + A)^*)$ denoting the monad of A-pointed idempotent semirings, HT-coalgebras are precisely context-free grammars in weak Greibach normal-form [5]. As the LFF of $\hat{H}$, we get precisely the context-free languages over A.

4.2 Recursive Program Schemes

In [2], recursive program schemes (rps) are characterized by f.p.-carried coalgebras for an endofunctor $\mathscr{H}$ on the base category $H_\Sigma/\mathsf{Mnd}_\mathsf{f}(\mathsf{Set})$ – finitary Set-monads T together together with a natural transformation $H_\Sigma \to T$, where H_Σ is the functor for the signature of “givens”. Furthermore, they characterize the solutions of all rps’s – the *algebraic* Σ-trees – as the union of all images of the corresponding coalgebras in the terminal $\mathscr{H}$-coalgebra.

We prove that this union is precisely the LFF of $\mathscr{H}$. Thereby we have solved the open problem of [2] to characterize algebraic Σ-trees by a universal property.

References

1 Jiří Adámek, Stefan Milius, and Jiří Velebil. Iterative Algebras at Work. *Mathematical Structures in Computer Science*, 16(6):1085–1131, 2006.

2 Jiří Adámek, Stefan Milius, and Jiří Velebil. On second-order iterative monads. *Theoretical Computer Science*, 412(38):4969 – 4988, 2011.

3 Sergey Goncharov, Stefan Milius, and Alexandra Silva. Towards a coalgebraic Chomsky hierarchy (extended abstract). In Josep Diaz, Ivan Lanese, and Davide Sangiorgi, editors, *Proc. 8th IFIP TC 1/WG 2.2 International Conference on Theoretical Computer Science (TCS'14)*, volume 8705 of *Lecture Notes Comput. Sci.*, pages 265–280. Springer, 2014.

4 Alexandra Silva, Filippo Bonchi, Marcello Bonsangue, and Jan Rutten. Generalizing determinization from automata to coalgebras. *Logical Methods in Computer Science*, 2013.

5 Joost Winter, Marcello Bonsangue, and Jan Rutten. Coalgebraic Characterizations of Context-Free Languages. *Logical Methods in Computer Science*, 9(3), September 2013.