Varieties, Quasivarieties and Prevarieties: Completing the Picture

Wataru Hino and Ichiro Hasuo[1]

1 **Department of Computer Science, The University of Tokyo**
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-8656, Japan
{wataru, ichiro}@is.s.u-tokyo.ac.jp

Characterization of (Quasi-)Varieties *Variety* and *quasivariety* of algebras are classic notions in universal algebra (see e.g. [3]). By definition, a variety is a full subcategory of $\mathbf{Alg}\Sigma$ specified by a set of equations; a quasivariety is specified by a set of *implications* $\forall\vec{x}\ \big((\bigwedge_{i=0}^{n} s_i = t_i) \to s = t\big)$. Then the famous Birkhoff theorem characterizes varieties as those which are closed under *homomorphic images*, *subobjects* and (arbitrary) *products* (H, S, P in Table 1). A similar characterization is possible for quasivarieties (see [3]): see Table 1, where FC means closure under *filtered colimits*.

These classic results are significantly extended through the development of categorical model theory (see e.g. [3]). This accounts for the rows of Table 1 other than the first and third rows.

▶ **Definition 1** (orthogonality [9]; see also [10])**.** Let us fix a category $\mathcal{A}$. Let $\mathcal{M}$ be a class of morphisms in $\mathcal{A}$; and $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

- A morphism $f\colon A \to B$ and an object C are *orthogonal* (we write $f \perp C$) when $(-)\circ f\colon \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ is bijective.
- $\mathcal{X}^{\perp}$ is the class of morphisms orthogonal to each object in $\mathcal{X}$.
- $\mathcal{X}$ is an $\mathcal{M}$*-orthogonality class* if there is a subclass $\mathcal{M}'$ of $\mathcal{M}$ such that $\mathcal{X} = \mathcal{M}'_{\perp}$. An *orthogonality class* is an $\mathcal{M}$-orthogonality class where $\mathcal{M}$ is the class of all morphisms.

In the second row of Table 1, a variety is characterized as an $\mathcal{M}$-orthogonality class where $\mathcal{M}$ is the class of regular epis $f\colon FX \twoheadrightarrow B$ from a free finitely presentable (FP) Σ-algebra FX to an FP Σ-algebra B. For quasivarieties the domain need not be free. In the fourth row of Table 1, furthermore, (quasi)varieties are identified with reflective subcategories of $\mathbf{Alg}\Sigma$ that are *epireflective* (meaning that reflections are regular epis) and are subject to additional closure requirements.

Besides these "extrinsic" characterizations, "intrinsic" ones are possible, too, that do not depend on the ambient category $\mathbf{Alg}\Sigma$. Varieties are Eilenberg-Moore categories for finitary

characterization		variety	*sort-of-variety*	quasivariety	prevariety [4]
extrinsic (in $\mathbf{Alg}\Sigma$)	logical (where $E \equiv \bigwedge \overrightarrow{s = t}$)	equations $s = t$	$\forall\exists!$-formulas $\forall\vec{x}\ \exists!\vec{y}\ E$	implications $\forall\vec{x}\ (E \to s = t)$	preequations $\forall\vec{x}\ (E \to \exists!\vec{y}\ E')$
	by orthogonality	$FX \twoheadrightarrow B$	$FX \to B$	$A \twoheadrightarrow B$	$A \to B$
	by closure property	H, S & P	?	S, P & FC	$\mathcal{A}$-pure S, P & FC
	by reflectivity	epirefl. & H	?	epirefl. & FC	reflective*
intrinsic	as a concrete cat.	(finitary) monadic	?	algebraic†	?
	generator	exactly proj.	?	regularly proj.	arbitrary†

Table 1 Characterizations of notions of variety. Here Σ is a finitary signature (i.e. every operation has a finite arity). In $*$ we have to allow infinitary conjunction in E, E', quantification over infinitely many variables, and an additional size constraint called *bounded generation* [4] is imposed. In † we allow classes of operations and equations, infinite arities and infinitary logic.

Conference title on which this volume is based on.
Editors: Billy Editor and Bill Editors; pp. 1–3
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

monads; the notion of algebraicity (for quasivarieties) is found in [2]. Finally, the bottom row of Table 1 presents characterization via generators [1].

Prevarieties Let us move on to the fourth column of Table 1. In [4] the notion of *prevariety* is introduced. It is motivated via the characterization by reflectivity (the fourth row): by dropping the epiness requirement we are led to prevarieties. It is shown in [4] that the notion has a nice logical characterization—see the top-right cell, where the quantifier $\exists!$ works much like the definition of extra partial/conditional operations. Furthermore, for each cardinal λ, it is shown that λ-ary prevarieties coincide with locally λ-presentable categories.

Just like the case of varieties and quasivarieties, (finitary) prevarieties can be characterized as $\mathcal{M}$-orthogonality classes where $\mathcal{M}$ is the class of morphisms between FP Σ-algebras. They also have characterizations by closure properties—as classes closed under $\mathcal{A}$-pure subobjects (defined in [4]), products and filtered colimits—and by generators. See [4].

Completing the Picture It is then natural to think of the intermediate notion—the second row of Table 1—that we shall take the liberty of calling *sort-of-variety.* It is obtained from prevarieties by prohibiting premises in preequations. Such $\forall\exists!$-*formulas* are understood to define extra *total* operations, unlike *partial* ones in prevarieties (cf. Proposition 4 later).

▶ **Definition 2.** A *sort-of-variety* is a full subcategory of $\mathbf{Alg}\Sigma$ defined by a set of $\forall\exists!$-*formulas* $\forall\vec{x}\ \exists!\vec{y}\ (\bigwedge_{i=0}^{n} s_i = t_i)$.

We are yet to investigate the nature of sort-of-varieties—as witnessed by many "?" in the second column of Table 1—let alone coming up with a proper name for the notion. In the rest of this abstract, nevertheless, we shall list some facts that we already know.

Let us fix a signature Σ; the known relationships between the four columns (when seen extrinsically as subcategories of $\mathbf{Alg}\Sigma$) are as follows. Varieties are sort-of-varieties, since every equation is a $\forall\exists!$-formula (with no variables bound by $\exists!$). The converse is not true; see Example 3.2. There are, too, sort-of-varieties that are not quasivarieties (Example 3.1).

▶ **Example 3.** **1.** Let $\Sigma = \{\cdot\}$ be the signature for semigroups. The class of *groups* can be specified by $\forall\exists!$-formulas, in $\mathbf{Alg}\Sigma$: associativity, $\forall x\ \forall y\ \exists! z\ (x \cdot z = y)$ and $\forall x\ \forall y\ \exists! z\ (z \cdot x = y)$. This class cannot be defined by implications in $\mathbf{Alg}\Sigma$ since it is not closed under subsemigroups.

2. Let $\Sigma = \{+, 0\}$ be the signature for monoids. The class of torsion-free abelian groups is a sort-of-variety in $\mathbf{Alg}\Sigma$. Indeed it is characterized by: the equations for commutative monoids; *invertibility* $\forall x\ \exists! y\ (x + y = 0)$; and *torsion-freeness* $\exists! x\ (n \cdot x = 0)$ for each $n \in \mathbb{N}_+$ where $n \cdot x$ stands for $x + \cdots + x$ (n times).

It follows from Def. 2 that a Σ-homomorphism preserves the extra operations introduced by $\forall\exists!$-formulas. This, together with Example 3.1, explains an elementary phenomenon in group theory that a map between groups that preserves multiplication is a group homomorphism.

Now let us investigate the relationships between the four columns of Table 1 when we *allow* to change a signature Σ. From this *intrinsic* viewpoint, sort-of-varieties are indeed quasivarieties.

▶ **Proposition 4.** *Let $\mathcal{A}$ be a full subcategory of $\mathbf{Alg}\Sigma$ specified by a set of* $\forall\exists!$-formulas. *Then there exist: a signature Σ' (such that $\Sigma' \supseteq \Sigma$), and a full subcategory $\mathcal{A}'$ of $\mathbf{Alg}\Sigma'$ that is specified by implications, such that the canonical forgetful functor $U\colon \mathbf{Alg}\Sigma' \to \mathbf{Alg}\Sigma$ restricts to an isomorphism $U'\colon \mathcal{A}' \to \mathcal{A}$.*

Proof. Let $\mathcal{E}$ be the set of $\forall\exists!$-formulas that defines $\mathcal{A}$. For each $\phi \equiv \forall\vec{x}\ \exists!\vec{y}\ E^{\phi}(\vec{x},\vec{y})$ in $\mathcal{E}$, where $\vec{x} = (x_i)_{i=1}^{n}$ and $\vec{y} = (y_j)_{j=1}^{m}$, we introduce m operations $\vec{f^{\phi}} = (f_j^{\phi})_{j=1}^{m}$ of arity n. Let $\Sigma' = \Sigma \cup \{f_j^{\phi} \mid \phi,\ j\}$, and define $\mathcal{E}'$ to be the collection of formulas $\forall\vec{x}\ E^{\phi}(\vec{x},\vec{f^{\phi}}(\vec{x}))$ together with $\forall\vec{x}\,\forall\vec{y}\ \big(E^{\phi}(\vec{x},\vec{y}) \to y_j = f_j^{\phi}(\vec{x})\big)$, for each ϕ and j. It is easily checked that U' is well-defined, bijective on objects and fully faithful. ◀

Finally we notice that sort-of-varieties allow a characterization by orthogonality. The proof is much like for prevarieties [4].

▶ **Proposition 5.** *Sort-of-varieties are precisely $\mathcal{M}$-orthogonality classes in* $\mathbf{Alg}\Sigma$ *where $\mathcal{M}$ is the class of morphisms from a free FP Σ-algebra to an FP Σ-algebra.*

Future Work Obviously we wish to complete Table 1, filling the "?" cells as well as getting rid of $*$ and † (i.e. discovering suitable finitariness conditions).

In Proposition 4 we showed that a sort-of-variety in $\mathbf{Alg}\Sigma$ can be seen as a quasivariety if we allow extension of the signature. Whether its inverse holds or not is open.

In fact, we arrived at the notion of sort-of-variety via our inspection of the works [7, 5, 6, 8] of Battenfeld and Schröder. These works are based on categories of topological spaces (unlike **Set** here); in [7, 8] they study so-called *observationally-induced algebras* that are characterized by the same orthogonality as for sort-of-varieties. The current abstract contributes, in **Set**: 1) their logical characterization in terms of $\forall\exists!$-formulas; and 2) putting the notion in a context of various variety notions. Taking Table 1 to a topological setting and investigating relationships to the results in [7, 5, 6, 8] is therefore obvious future work.

Acknowledgments Thanks are due to Soichiro Fujii, Toshiki Kataoka and Hiroshi Ogawa for helpful discussions. The authors are supported by Grants-in-Aid No. 24680001 & 15K11984, JSPS.

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